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Post-buckling analysis of planar elastica using a hybrid numerical strategy

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ABSTRACT

Many structural mechanics problems, such as post-buckling of elastica, involve determining multi-equilibrium states of a nonlinear system. Typically, the stable equilibrium states are found by searching for minima of the potential energy function subjected to some constraints. In this paper, the method of genetic algorithm combined with the quasi-Newton method is applied to search for the multiple minima of the strain energy of various elastica problems. The proposed hybrid methodology is relatively straightforward to implement and is adopted to study post-buckling behaviour of planar elastica modelled as multi-link systems for various boundary conditions and sidewall constraints.

1. Introduction

In structural mechanics, many nonlinear problems such as elastica and post-buckling of elastic structures involve a category of geometrical nonlinear and material elastic systems. Motivated by the development of long-span space structures and non-terrain based deployable structures, interests in understanding and modelling the large displacement behaviour of flexible structures have grown. Thompson and Hunt presented a comparative study among continuum analysis, discrete system analysis and finite element analysis [1]. Murakawa et al. utilized the complementary energy principle and an associated hybrid finite element method to analyze stability of structures [2]. Anifantis and Dimarogonas studied the post-buckling behaviour of transverse cracked column [3]. In their study the buckling model of the cracked column is represented using a flexible rotational spring (cracked section) to join two uncracked segments. Lately, nanotubes and DNA molecules have also been represented by structural models with geometrical nonlinearity and material elasticity [4,5]. All the above stated structures can be modelled as multi-link systems. For such nonlinear elastic problems, static equilibrium configuration of the system is one that possesses stationary strain energy (local minimum), as stated in Bernoulli's principle. Since the system is nonlinear, multiple configurations corresponding to different energy levels can exist under the same geometric constraints. Searching and identification of multiple configurations is often necessary for better understanding of the system.

Nevertheless, conventional Newton–Raphson iteration techniques face numerical difficulties especially when the equilibrium path approaches the limit point. To study post-buckling behaviour beyond the limit point, Riks proposed an analysis technique (known as Riks method or the arc length method) to search the equilibrium path in the load–displacement space [6]. This technique has been successfully implemented in the finite element method [7–10]. The arc length method defines load as an additional unknown that can be solved by incorporating a constraint in terms of displacement in the previous step, the proportional load factor, and the arc length. Several early implementations in the finite element method incorporating arc length constraints include tangent plane arc length [6] and spherical constant arc length [8], and if necessary, constant increment of external work method may be used as well [7,9]. A usual requirement for post-buckling analysis by the arc length method is the introduction of imperfection, which can be obtained by linear eigenvalue buckling analysis. In this way, bifurcation point can be passed to provide a continuous searching space for a particular buckling mode. The arc length method, however, cannot directly obtain a solution for a specified load or displacement as they are treated as unknown during the solution procedure. For general post-buckling problems involving complex contact conditions, the application of the arc length is not straightforward [10] though there have been successful attempts in solving specific contact problems [11,12]. The advantages and drawbacks of the arc length method have been discussed in Refs. [13,14]. Against this backdrop, an alternative approach is explored in this paper.

A hybrid numerical strategy is proposed combining genetic algorithm (GA) for constrained minimization problems, which was proposed by Moerder and Pamadi [15], and quasi-Newton

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Nomenclature

a	distance between two ends of elastica
b	parameter defining the stiffness characteristic of sidewall
C	a user-defined penalty weight
d_1, d_2	distances from sidewalls to x axis
D	displacement of the moving end in x direction
E	Young's modulus
I	moment of inertia of the cross-section
K_i	spring constant of elastic rotational spring connecting
L	total length of elastica
$\mathscr{L}$	Lagrangian function
M	moment
P_{cr}	critical Euler buckling load
S_i	ith segment length
SE	strain energy
U	objective function
w	maximum deflection
ψ_i	relative angle of adjacent two segments in Model 1, and slope at the ith node in Model 2
$\boldsymbol{\psi}^*$	local minimum
λ_1	Lagrange multiplier, which equals reaction force in x direction
λ_2	Lagrange multiplier, which equals reaction force in y direction

method to study the large displacement behaviour of various multi-link systems. There is no premise for introduction of buckling mode. The hybrid strategy is illustrated in the context of planar elastica. Numerical results are also compared with the analytical solutions available for uniform elastica [16,17].

While many past researchers employed analytical elliptic integral or numerical shooting method, the numerical strategy presented in this paper is relatively straightforward and easy for implementation to study the post-buckling behaviour of elastic system subjected to different boundary conditions and constraints. Pin–pin elastica, clamp–pin elastica, clamp–clamp elastica, pin–pin elastica subjected to two rigid frictionless sidewalls and pin–pin elastica with transverse cracked section are analyzed using the proposed strategy. In addition, the snap through of a hinged right-angle frame subject to fixed point load, which has been studied by Argyris et al. [18] and Simo et al. [19], is modelled with this strategy.

2. Genetic algorithm and quasi-Newton method

In recent years, the use of GA has gained popularity in many fields including structural engineering. GA embraces the doctrine of survival of the fittest and has two major operators, *i.e.* crossover and mutation which are keys to the natural evolution and selection in the biological world. This probabilistic population-based search strategy has the advantage of having high likelihood of finding the global minimum. However, GA is generally considered suitable for unconstraint minimization problems. For constraint satisfying problems (CSP), which are common in many engineering problems, a key issue is to employ the constraint function to enforce the contact conditions [20]. One common approach is to convert the original problem into an unconstrained one by constructing a weighted penalty function, or a Lagrangian function, or both. The penalty can be constant or adaptive. If a constant penalty weight is used, the magnitude of the penalty can significantly affect the search procedure. If the weight is too "heavy", many individuals in the mating pool that are not strictly compatible with the constraints will become extinct too early in the GA selection procedure. The population will thus lose its diversity resulting in premature convergence to a local optimal point. If the penalty is too "light", it will be computationally inefficient to converge to the desired constrained solution. Therefore, the choice of weight largely depends on the problem and user's experience, and re-tuning of the weight is often inevitable. In this paper, an adaptive penalty function is utilized. The fitness evaluation function is composed of the gradient of the Lagrange function and the constraints. Lagrangian multipliers are evaluated from the least square condition of the local minimum [15].

Due to its stochastic nature, the results of GA search are often not guaranteed for problems with a large number of unknowns and nonlinear constraints. The difficulty is twofold. First, the search with GA alone is not effective to yield good result when there are many unknowns. Second, if high resolution of the search space is required to produce accurate search, the computational cost will be prohibitive. On the other hand, the quasi-Newton method based on gradient information of the problem is efficient in finding local minimum provided that a promising initial guess is presented. We combine these two methods in our search for the multiple local minima of CSP. To this end, penalty functions and a memory list are employed to propel the searching procedure away from the neighbourhood of previously obtained solutions. This is necessary to provide promising candidates in the neighbourhood of a different local minimum for the quasi-Newton method to continue the search. The quasi-Newton method updates the variables and multipliers simultaneously, and BFGS (Broyden, Fletcher, Goldfarb, and Shanno) formula is employed to approximate the evaluation of the inversion of Hessian matrix [21].

Lagrangian function has the following form:

$$\mathscr{L}(\psi,\lambda) = U(\psi) - h\lambda \tag{1}$$

where U, ψ, h, λ are the objective function, variables, equality constraints and Lagrange multipliers, respectively. The necessary condition for a local minimum is that the first order gradient of Lagrange function at a local minimum equals to zero, i.e.

$$\nabla_\psi \mathscr{L}(\psi^*,\lambda^*) = U_\psi(\psi^*) - h_\psi(\psi^*)\lambda = 0 \tag{2}$$

where subscript indicates differentiation, and ψ^* and λ^* are the local optimum and corresponding Lagrange multipliers. In this approach, the Lagrange multipliers are not treated as independent variables. From Eq. (2), we can estimate the value of Lagrange multiplier as:

$$\begin{pmatrix} \lambda_1 \\ \lambda_2 \\ \vdots \\ \lambda_m \end{pmatrix} = \begin{pmatrix} h_{1\psi_0} & h_{2\psi_0} & \dots & h_{m\psi_0} \\ h_{1\psi_1} & h_{2\psi_1} & \dots & h_{m\psi_1} \\ \vdots & \vdots & \ddots & \vdots \\ h_{1\psi_{n+1}} & h_{2\psi_{n+1}} & \dots & h_{m\psi_{n+1}} \end{pmatrix}^{\dagger} \begin{pmatrix} U_{\psi_0} \\ U_{\psi_1} \\ \vdots \\ U_{\psi_{n+1}} \end{pmatrix} \tag{3}$$

where $h_j(\psi)(j = 1,\dots,m)$ is one of the equality constraints of the problem, and m is the number of equality constraints. The operator † denotes Moore–Penrose inversion or pseudo-inversion defined as $h_\psi^\dagger = (h_\psi^T h_\psi)^{-1} h_\psi^T$.