Physica A 485 (2017) 61–72

Contents lists available at ScienceDirect

Physica A

journal homepage: www.elsevier.com/locate/physa

Low-traffic limit and first-passage times for a simple model of the continuous double auction

Enrico Scalas [a,b,*], Fabio Rapallo [c], Tijana Radivojević [b]

[a] Department of Mathematics, School of Mathematical and Physical Sciences, University of Sussex, Falmer Campus, BN1 9QH, Brighton, UK
[b] BCAM, Basque Center for Applied Mathematics, Alameda de Mazarredo 14, 48009 Bilbao, Basque Country, Spain
[c] DISIT, Università del Piemonte Orientale, viale Teresa Michel 11, 15121 Alessandria, Italy

H I G H L I G H T S

- We consider a model of the continuous double auction.
- We study this model in the so-called low-traffic limit.
- We find an exact expression for the price distribution in the low-traffic limit.
- We study the first-passage time in 1 or N.

A R T I C L E   I N F O

Article history:
Received 2 March 2017
Received in revised form 12 May 2017
Available online 20 May 2017

Keywords:
Double auction
Markov chains
Queueing theory
Low-traffic limit

A B S T R A C T

We consider a simplified model of the continuous double auction where prices are integers varying from 1 to N with limit orders and market orders, but quantity per order limited to a single share. For this model, the order process is equivalent to two $M/M/1$ queues. We study the behavior of the auction in the low-traffic limit where limit orders are immediately matched by market orders. In this limit, the distribution of prices can be computed exactly and gives a reasonable approximation of the price distribution when the ratio between the rate of order arrivals and the rate of order executions is below $1/2$. This is further confirmed by the analysis of the first-passage time in 1 or N.

1. Introduction and model description

Most of the regulated markets in the world implement a trading mechanism known as the *continuous double auction* to match supply and demand. This mechanism has two sides. On the supply side there are orders to sell and on the demand side there are orders to buy. Hence, the auction is called *double*. Moreover it occurs in continuous time. Hence, it is called *continuous*.

To be more specific, in, say, a regulated equity market, a continuous double auction consists of a *book* for each share where orders to buy (*bids*) and to sell (*asks*) are recorded. An order consists of a price at which the seller (buyer) wants to sell (buy), of a quantity of shares they want to sell (buy) and of further specifications, for instance concerning order cancellation. The possible order types are described by market regulations. Orders can be sent directly to the book and, in this case, we have an *order-driven* market. On the contrary, if only bids and asks sent by market makers and/or specialists are accepted, we

* Corresponding author at: Department of Mathematics, School of Mathematical and Physical Sciences, University of Sussex, Falmer Campus, BN1 9QH, Brighton, UK.
E-mail addresses: e.scalas@sussex.ac.uk (E. Scalas), fabio.rapallo@uniupo.it (F. Rapallo), tradivojevic@bcamath.org (T. Radivojević).

http://dx.doi.org/10.1016/j.physa.2017.05.020
0378-4371/

have a so-called *quote-driven* market (see [1] for a classical introduction to the topic and [2] for the new problems related to high-frequency trading). For our purposes, it is important to distinguish between two main order types: *market orders* and *limit orders*. A market order is an order where the seller (buyer) accepts to sell (buy) a certain quantity of shares at the best available price. In other words, the seller (buyer) accepts the best bid (ask) present in the book. When a market order is executed a transaction takes place and shares are exchanged. A limit order is an order to buy a certain quantity of shares at a price not higher than a given value or to sell a certain quantity of shares at a price not lower than a given value. Typically, limit orders are stored in the book until they are totally or partially executed or cancelled after some time if they are not executed.

In recent years, the theory of this auction has gained more and more interest. Despite all the possible complications of real continuous double auctions, their theory can be based on queueing theory. In fact, we can consider the book as a peculiar queue and orders as *customers* that wait to be served (executed). In particular, it has been shown that appropriate models of the double auction can be mapped into a multi-class queue [3], so that its ergodic properties and the limiting invariant distribution can be studied using established techniques [4].

In this paper, we consider a simplified model (see [5] and references therein) where prices take N integer values from 1 to N. Only two types of orders are considered: *limit orders* and *market orders*. In their turn, limit orders can be either orders to sell a single share at a price not lower than a given amount (asks) or orders to buy a single share at a price not higher than a given amount (bids). In other words, the quantity attached to every limit order is always 1. Among all the asks, the *best ask* is the smallest ask price, whereas the *best bid* is the largest bid price. The best bid is always strictly smaller than the best ask. Market orders have also two sides: either they accept the available best bid or the available best ask. For the sake of simplicity, limit ask orders and limit bid orders arrive according to a Poisson process at a rate λ_a and λ_b, respectively. In the following, we assume symmetry, i.e. $\lambda_a = \lambda_b = \lambda$. Market orders to buy and market orders to sell arrive separated by durations following the exponential distribution with parameter μ_b and μ_a, respectively. Again, symmetry is assumed, namely $\mu_a = \mu_b = \mu$. Limit ask orders follow the uniform distribution in the interval from $p_b + 1$ to $p_b + n$, where p_b is the current best bid and $n \geq 1$ is a parameter of the model. Similarly, limit bid orders are uniformly drawn from the interval $p_a - n$ to $p_a - 1$, where p_a is the current best ask. The accessible states of the auction are limited by the condition $p_b < p_a$. When p_a is between 1 and n (p_b between $N - n + 1$ and N), the bid (respectively ask) interval is restricted correspondingly. For instance, if $p_a = 1$, bids are impossible. The parameter n acts as a cut-off for price jumps. Eventually, if no orders are present in the auction, the next bid, b, is uniformly chosen in the interval $p - n \leq b \leq p$ and the next ask, a, is uniformly taken from $p \leq a \leq p + n$, where p is the price of the last trade. Specifying an initial price (the opening auction price) is sufficient to start the auction. A short remark is necessary at this stage: It turns out that order inter-arrival times are not exponentially distributed in real markets (see [6] and references therein). This means that the above description in terms of $M/M/1$ processes should be replaced by a semi-Markov description in terms of $G/G/1$ processes. However, in this paper, for the sake of simplicity, we will limit our analysis to the Markovian case.

The model described above is essentially the same as in [7] and in [8]. It is a *zero-intelligence agent-based model* [9]. As already mentioned in [5], this version of the model does not use the uniform distribution over $[0, \infty)$ as in [7] and it is not limited to the case in which limit orders arrive only at the best bid/ask price as in [8]. A preliminary discussion of this version was presented in [10]. This model was extensively studied in [11], in the case in which price movements equal one tick. These authors also studied the heavy-traffic limit [12] where functional limit theorems are available leading to diffusion approximations [13–15].

This class of simple models can also help in clarifying the relationship between market structure and agents behavior, beyond the zero-intelligence limit [16,17].

In [5], the focus was on the ergodic properties of the model. Based on the fact that the order process is equivalent to two independent $M/M/1$ queues, it was shown that there are three regimes depending on the value of $\rho = \lambda/\mu$. For $0 < \rho < 1$, prices are free to fluctuate over the full price range and statistical equilibrium is reached (ergodic regime). For $\rho \geq 1$, the auction is in a non-ergodic regime which stabilizes prices. Due to the presence of the parameter n, there is an additional transition. If $1 \leq \rho < n$, prices can still fluctuate in a limited range, whereas for $\rho \geq n$, prices eventually fluctuate between two values. This regime cannot be found if one only considers the case $n = 1$. It is useful to remark that ρ is the relevant parameter in this model. Given that ρ is the ratio between λ and μ, the same value of ρ can be obtained with different values of these rates.

In the following, we further characterize the ergodic regime by considering the so-called *low-traffic* limit where $\rho \ll 1$. It is a limit where analytic results are available for the price dynamics as discussed below. Moreover, we study the first-passage time of the auction in 1 or in N. The study of the first-passage time is useful to characterize the price stochastic process. It turns out that this analysis provides useful approximations for the behavior of the auction model when $\rho < 1/2$. In the spirit of statistical physics, in the following, we study what happens in the limit of vanishing ρ, a situation where an exact solution for the price dynamics is available. In the last section of the paper, we further discuss the meaning of this limit.

It is useful to explicitly show a strict analogy between what we are doing in this paper and what is usually done in statistical physics. Even if the continuous double auction is a *social construct*,[1] once it is put into place, such a construct becomes part of nature, at least temporarily,[2] and prone to scientific investigation. In particular, it becomes possible and

[1] The Oxford Dictionary defines a *social construct* as a "social phenomenon or convention originating within and cultivated by society or a particular social group, as opposed to existing inherently or naturally".

[2] Without presenting a detailed history of auctions, we can state with some confidence that there has been a long period in human history before the introduction of the continuous double auction. And, perhaps with less confidence, we can state that there will be a more or less remote future without continuous double auction.