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Ultimate state of plane frame structures with piecewise linear yield conditions and multi-linear behavior: A reduced complementarity approach

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ABSTRACT

Elastoplastic analysis of structures with mathematical programming methods aims at finding the load factor of a given load pattern subject to equilibrium and compatibility requirements, satisfying yield and complementarity constraints. A new approach is introduced that identifies the specific yield hyperplanes associated with all critical sections avoiding all irrelevant alternatives. This results into substantial reduction of the size of the yield and complementarity conditions. In addition, it has a beneficial effect in addressing multi-linear hardening and/or softening holonomic behavior by controlling the size of the problem. Numerical examples are presented that verify the efficiency of the proposed approach.

1. Introduction

Limit analysis of structures based on rigid-perfectly plastic constitutive behavior has offered the means to assess the ultimate capacity of frame, plate and other structures. Static-safe and kinematic theorems proved very efficient not only in calculating the ultimate state, but also in changing the mentality in the conceptual design of structures as well as structural components and connections.

The books of Massonet and Save [1], Neal [2] and others laid a solid base for this development. The mathematical formulation of limit analysis within the realm of linear programming secured further the establishment of this methodology. A decisive step forward was made by Maier and his coworkers [3–7] that extended the formulation to account for isotropic and kinematic hardening/softening behavior addressing both holonomic and non-holonomic problems. The ultimate load-carrying capacity of the structure is determined by solving an optimization problem with equilibrium, compatibility, yield and complementarity constraints. The whole formulation is based on multi-linear constitutive relations following plastic deformation theories [8].

This enhancement was driven and supported from the developments in mathematical programming that treated properly complementarity problems. More specifically, the exploration of the complementarity problem by Cottle [9] has directed the formulation of elastoplastic analysis in the form of a Parametric Linear Complementarity Problem [6], while Kaneko later proposed a reformulation of this problem [10]. Moreover, a variety of alternative mathematical programming procedures have been applied for elastoplastic analysis of structures such as Iterative Linear Programming, Quadratic Programming, Restricted Basis Linear Programming, Parametric Linear Complementarity and Parametric Quadratic Programming [3,5,6,11,12].

The recent development of algorithms appropriate for Mathematical Programming with Equilibrium Constraints (MPEC) problems [13–15] has extended the potential of the proposed methods for structural analysis for both holonomic and nonholonomic assumptions. Extensive work in this direction was conducted by Tin-Loi and coworkers [16–21]. More recently, in this context, softening structural behavior was also examined under the effect of interaction (axial force-bending moment) by Cocchetti and Maier [22], Ardito et al. [23] and Tangaramvong and Tin-Loi [24,25].

The standard formulation of elastoplastic analysis with mathematical programming is based on the notion of calculating the strength reserves for every critical section and for all possible segments of the piecewise linear (PWL) yield surface. This defines a vector of reserves for every critical section with multiplicity equal to the number of segments of the yield surface. For a more general interaction, this evaluation is extended to every hyperplane increasing the dimension of the vector of strength reserves accordingly. The same number of plastic multipliers is engaged also for all possible plastic deformations, which together with the corresponding strength reserves compete within the discrete in nature complementarity condition. This perplexing procedure generates

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unnecessary information that increases prohibitively the size of the problem especially for a finer piecewise linear discretization of the yield surface.

This work aims at reducing the size and complexity of the current formulation addressing the elastoplastic problem within the framework of mathematical programming. More specifically, the proposed simplification is threefold and concerns the evaluation of strength reserves, the direct evaluation of the hardening/softening response at every critical section and the reduced formulation of the complementarity condition. The main step towards these goals is the identification of the particular cone in which every stress vector resides for all critical sections at every loading stage of the optimization process. Based on this information, both the reserves and the complementarity condition of all critical sections are solely formulated for one particular cone per critical section and optimization step. Moreover, detection of the unique hardening/softening branch at which a critical cross-section is stressed, significantly simplifies the incorporation of multi-linear hardening law into the formulation of the problem. This avoids generation of all possible relations in every direction around the yield surface for all different hardening/softening branches in the constitutive relation.

The remaining sections are organized as follows. First, the governing relations that describe the holonomic elastoplastic problem are presented. Equilibrium, kinematic and constitutive relations together with yield and complementarity conditions that incorporate the concept of cone and hardening/softening branch identification are discussed. Then, the formulation of elastoplastic analysis as a MPEC problem incorporating the above concepts is presented. Subsequently, numerical results of plane steel frames are presented that illustrate the applicability and the efficiency of the proposed scheme.

2. Problem formulation

The elastoplastic analysis problem is defined on the basis of equilibrium, constitutive and kinematic equations, together with yield and complementarity conditions.

Plane frames are considered herein consisting of prismatic elements subjected only to nodal loading for simplicity reasons. Moreover, small displacements are assumed to establish equilibrium equations at the initial undeformed configuration. In addition, plastic behavior, if present, is considered only at preselected critical sections, i.e. the end sections of the elements, whereas the remaining parts behave elastically. The nonlinear inelastic behavior at critical sections is described by a multi-linear model and yield conditions are appropriately linearized. Furthermore, under the external loading, if local unloading occurs, is assumed happening along the load displacement path and not as elastic unloading, adopting a holonomic, i.e. path-independent structural behavior. Although this is a simplified assumption, especially for the case of softening behavior, it can be considered reasonable for monotonically increasing external actions [6,22,24]. Moreover, isotropic hardening is adopted, which under holonomic assumption and monotonic loading yields satisfactory results. For cyclic loading though, kinematic hardening is more appropriate and definitively closer to real behavior of steel structures.

The formulation of the problem requires treatment at three different levels, i.e. the level of critical cross sections, the element level and the structural level. All final equations are expressed in dimensional form at the structural level. The yield conditions, though, are first introduced in nondimensional form. Moreover, the method follows the sign convention of matrix structural analysis, whereas final results are presented on the basis of engineering sign convention.

2.1. Equilibrium

Each plane beam element develops six stress resultants at its ends, as shown in Fig. 1. Herein, the axial force (s_1^i), bending moment at the start node j (s_2^i) and bending moment at the end node k (s_3^i), are considered as independent primary actions for member i [25]. Thus the six end actions of the element can be expressed at the global axes system in terms of the local basic actions by using the corresponding equilibrium matrix as follows:

$$\begin{Bmatrix} F_x^j \\ F_y^j \\ M^j \\ F_x^k \\ F_y^k \\ M^k \end{Bmatrix} = \begin{bmatrix} \cos\omega^i & -\sin\omega^i/L^i & -\sin\omega^i/L^i \\ \sin\omega^i & \cos\omega^i/L^i & \cos\omega^i/L^i \\ 0 & 1 & 0 \\ -\cos\omega^i & \sin\omega^i/L^i & \sin\omega^i/L^i \\ -\sin\omega^i & -\cos\omega^i/L^i & -\cos\omega^i/L^i \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{Bmatrix} s_1^i \\ s_2^i \\ s_3^i \end{Bmatrix} = [B^i] \cdot \{s^i\} \tag{1}$$

where $F_x^j,\ F_y^j,\ M^j$ are the global X and global Y forces and bending moment at the start node and $F_x^k,\ F_y^k,\ M^k$ are the actions at the end node of the element i at the global system, ω^i is the angle formed rotating the global X-axis counterclockwise to meet the local x-axis and L^i is the element length, $[B^i]$ is the (6×3) equilibrium matrix of the element and $\{s^i\}$ is the (3×1) stress vector of the element.

The equilibrium for the whole structure is then established in terms of the unknown vector of primary actions of all members as:

$$[B] \cdot \{s\} = a \cdot \{f\} \tag{2}$$

where $[B]$ is the $(nf \times 3n)$ structural equilibrium matrix, assembled by the corresponding element equilibrium matrices arranged diagonally, $\{s\}$ is a $(3n \times 1)$ vector of all primary actions in local systems, a is a scalar load factor, $\{f\}$ the $(nf \times 1)$ vector of nodal loading in the global system, n denotes the number of elements and nf the number of degrees of freedom.

2.2. Compatibility

In Fig. 2 the initial-undeformed and the final-deformed configuration of a beam element with the corresponding nodal displacements and deformations are presented. For small displacements considered in this work, the relation between the member deformation $\{q^i\}$ in the local system and the nodal displacements $\{u^i\}$ at global axes system is given as:

$$\{q^i\} = [B^i]^T \cdot \{u^i\} \tag{3}$$

where $\{q^i\} = \{q_1^i \quad q_2^i \quad q_3^i\}^T$, q_1^i and q_2^i are the axial deformation and the rotation of the chord at the start node j and q_3^i is the rotation of the chord at the end node k of the member, $\{u^i\} = \{\mathrm{u}^j \quad \mathrm{v}^j \quad \theta^j \quad \mathrm{u}^k \quad \mathrm{v}^k \quad \theta^k\}^T$ is the vector of nodal displacements expressed at the global coordinate system containing the global X and global Y displacements and rotations of start node j and end node k respectively, while in Fig. 2 the corresponding hat quantities refer to local axes system. The (3×1) vector $\{q^i\}$ determines directly the deformation state of the element and dictates the selection of the primary end actions in the equilibrium relation (2).

The compatibility condition for the whole structure is then given by the following linear compatibility relation:

$$\{q\} = [B]^T \cdot \{u\} \tag{4}$$

where $\{q\}$ is the $(3n \times 1)$ deformation vector of the structure and $\{u\}$ is the $(nf \times 1)$ nodal displacement vector.