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# Large deflection of 3D curved rods: An objective formulation with principal axes transformations

M. Yilmaz *, M.H. Omurtag

*Faculty of Civil Engineering, Istanbul Technical University, Maslak, 34469 Istanbul, Turkey*

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ABSTRACT

This study shows that the deformed configuration space of 3D rods can be spanned in an objective manner by means of a semi-configuration-dependent approach. We employ a partially dependent default rotational triad defined using the spatial reference curve, a custom director-unit-vector, and a twisting-like-angle interpolation to obtain the true orientations of the cross-sections. Numerical verification shows that, rotating the custom director-unit-vector in conjunction with the element's first node, strain measure objectivity is ensured throughout the element domain. Employing the fundamental theorem of calculus, this simple but accurate finite-element implementation need not invoke the virtual work principle in the classical sense.



## 1. Introduction

Geometrically non-linear analysis of 3D framed structures has received considerable research attention. Among the various non-linear beam theories, Reissner's formulation [1] in the plane-frame case inspired other researchers. The theory was later extended to three dimensions with valuable contributions to its finite-element implementation by numerous authors [2–8]. Beam models of this type have been coined "geometrically exact" because, although the formulations assume non-deformability of cross-sections [9], the relationships between the configuration and the strain measures are consistent with the virtual work principle and the equilibrium equations at a deformed state regardless of the magnitude of displacements, rotations, and strains. A distinctive feature of this approach is that the kinematic variables at every point of the model include both displacements and rotations; hence it can be classified as a Cosserat theory [10]. Within the context of the theory, rotation parameters have been chosen in various ways [11–14]. Ibrahimbegovic [12] investigated the possibility of selecting the parameters for finite rotation representations from nine-parameter intrinsic orthogonal tensor representation to three-parameter representation using the so-called rotation vector. Jelenić and Saje [14] proposed a generalized form of the principle of virtual work by forcing the exact kinematic equations using a procedure similar to that of Lagrangian multipliers while eliminating the displacement variables of the model and retaining only rotational degrees of freedom. As they belong to a nonlinear manifold, 3D rotation variables must be chosen and treated carefully in order to maintain the characteristics of the rotation field during non-additive updates made in the context of the finite element method. Crisfield and Jelenić [9] showed that interpolation of the total rotational vector makes the resulting strain measures dependent on rigid-body rotation and proposed an objective setup based on interpolation of relative rotations. Their discussion paved the way for increasing attention to objectivity [15] and several alternative objective formulations [10,16–18] were later proposed.

A kinematic hypothesis in conjunction with the virtual work principle is considered a prerequisite in the studies mentioned above. On the other hand in [19], Smoleński argued that knowledge of 3D kinematics is not necessary for construction of the global theory of spatial rods. He successfully employed the definition of the standard definite-integration algorithm to extract the rod theory's general equilibrium equation from the 3D continuum balance laws and proposed a formulation by averaging these equations over the problem's domain. At the same time, he suggested that the only place for approximations in the theory of rods is in the constitutive relations.

There exist some other popular formulations to solve geometrically nonlinear beam problems. The co-rotational approach, viewed as an alternative way of deriving efficient non-linear finite elements for problems with large displacements but small strains, has garnered increased interest [20–26]. The main idea in this context is separation of rigid-body and purely deformational motion of the element through the use of a following frame which

* Corresponding author.
*E-mail addresses:* yilmazmura@itu.edu.tr (M. Yilmaz), omurtagm@itu.edu.tr (M.H. Omurtag).

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continuously rotates and translates with the element. The deformational response is captured at the level of this frame, whereas the geometric nonlinearity induced by large rigid-body motion is incorporated in the transformation matrices relating local and global internal force vectors and tangent stiffness matrices. Assuming the pure deformation part to be small, a geometric linear theory can be used in the local system. It is also shown in [21,22] that the co-rotational technique does not necessarily suffer from the disadvantages of using a finite rotational transformation.

In order to avoid rotational freedoms, an absolute nodal coordinate formulation (ANCF) was developed by Shabana [27,28] using absolute nodal position and slope degrees of freedom for the interpolation of the position field of beam elements; this uncovered 24 nodal degrees of freedom instead of 12 using a conventional two-node beam element. Other rotation-free methods [29,30] are also used to overcome the disadvantages of finite rotations, such as an unconventional finite element method in which rotations are not used as degrees of freedom and the element interpolation domains overlap.

Jonker and Meijaard [31] defined a set of independent, discrete deformation modes (DDM), related to conventional small-deflection beam theory, in a co-rotational frame by including geometric nonlinearities as additional second-order terms. This allowed them to express their influence on displacement, especially for bifurcation points where the load–deflection characteristics change drastically.

The formulation presented in this study is differs in several ways from the formulations summarized above. Initially, it bears resemblance to the co-rotational formulation; however, it is a different approach in the sense that both material and spatial reference curves are interpolated directly within their own attached local coordinate systems (ALCS) through their projections onto the ALCS planes, and deformations relative to the spatial ALCS are not restricted to be small. Expected accuracy for a coarse mesh is strictly correlated with parameterization and interpolation of both material and spatial configurations. An initially curved beam formulation helps greatly in this regard. Moreover, following the beam, the ALCS improves the interpolation capacity of the reference curve for representing the equilibrium configuration of the system with a relatively small number of elements. The spatial ALCS is defined in terms of nodal translations and total rotation of the element's first node, and it is deliberately selected to maintain objectivity in the discretized domain.

The presented formulation also shares a similar formalism with geometrically exact beam theory except that the effects of shear rotations are not accounted for and the virtual-work equations are exchanged for equivalent equations derived from the fundamental theorem of calculus. In so doing, we eliminate the use of virtual terms and the integration of stress resultants within the element domain, leading to very basic structure in the equilibrium equations. In this context, the present formulation has more features in common with the formulation presented in [19] than with any other. Our treatment of the finite rotations is also somewhat unique. From our point of view, the underlying cause of non-objectivity is the independent interpolation of the translation and the rotation fields. Instead of directly interpolating the total rotation vector, we transform the end-point principle axes using the Rodrigues' rotation formula and extract the local slopes of the spatial reference curve in conjunction with the spatial ALCS. In this way, we cover all dependencies between the translation and rotation fields of the spatial domain. Using the reference curve's tangent vector field, we propose a custom triad and a custom twisting-like angle, which serves as a rotational freedom, to find the true configuration of the cross sections. As the proposed angular freedom is the only rotational variable independent from the translational field, we have safely adapted a linear interpolation to this custom field. One of the prominent features of the proposed formulation is that it does not require the extraction of rotational parameters from the rotation matrix, unlike the method in [9]. Finally, we present the equations for evaluating the nodal values of the proposed angle in terms of the nodal rotation vector. Since the proposed formulation relies firmly on both the nodal and internal cross-sections' transformations, we deem it suitable to name it the "principal axes transformation" (PAT) formulation.

An outline of the rest of the article is as follows. Section 2 summarizes the definition of a custom rotational field, including the Frenet–Serret (FS) frame, when provided the tangent vector field of a space curve. The section also exposes some drawbacks of the FS frame in a simple fashion. In Section 3, the parameterization of the material configuration is covered in detail. Section 4 presents the parameterization of the spatial domain and the expression for finding the actual cross-section orientations by means of the twisting-like angle. This section also exhibits the way we constructed the finite-element setup. The next three sections describe the constitutive model, objectivity verification of the strain measures, and a summary of the solution algorithm, respectively. In Section 8, the accuracy and rate of convergence of the proposed formulation are investigated in several examples with different characteristics, such as large deflection of a cantilever rod, large deflection of an arc-shaped rod with mixed boundary conditions subjected to distributed loading, and a deployable circular ring problem. The article is concluded with a summary and discussion of the possible extension of the presented formulation in Section 9.

## 2. Default rotational field

Before presenting the proposed formulation, we consider it necessary to express the way we construct our custom orthonormal triad: $\mathbf{A}_i \in \mathrm{R}^3,\ i = 1,2,3$, related to the Cartesian frame bases $\mathbf{E}_i$ through a transformation $\Theta \in SO(3,\mathrm{R})$ as $\mathbf{A}_i = \Theta\mathbf{E}_i$, where $SO(3,\mathrm{R})$ is the 3D rotation group of proper orthogonal transformations. Strictly speaking, $\Theta$ is a $3 \times 3$ rotation matrix with its columns being the column vectors $\mathbf{A}_i$ arranged in the form: $\Theta = [\mathbf{A}_1, \mathbf{A}_2, \mathbf{A}_3]$. Given any vector $\mathbf{A}_1 \in \mathrm{R}^3$ with $\|\mathbf{A}_1\| = 1$, the space of $\Theta(\mathbf{A}_1, \mathbf{A}_2, \mathbf{A}_3)$ reduces to $\Theta(\mathbf{A}_1, \mathbf{e})$ as in

$$\Theta(\mathbf{A}_1, \mathbf{e}) = \left[\mathbf{A}_1, \hat{\mathbf{A}}_1^{\mathrm{T}}\mathbf{A}_{1e}, \mathbf{A}_{1e}\right] \tag{1}$$

where $\mathbf{e} \in \mathrm{R}^3$ is an arbitrary unit vector denoting an arbitrary direction with the restriction $\mathbf{e}$ is not parallel to $\mathbf{A}_1$. In Eq. (1), $\mathbf{A}_{1e}$ is also a unit vector defined with cross-product operations as

$$\mathbf{A}_{1e} = \mathbf{A}_1 \times \mathbf{e}/\|\mathbf{A}_1 \times \mathbf{e}\| = \hat{\mathbf{A}}_1\mathbf{e}/\|\hat{\mathbf{A}}_1\mathbf{e}\| \tag{2}$$

which ensures that $\mathbf{A}_{1e} \perp \mathbf{A}_1$. In the equations above and throughout the text, the hat denotes a skew-symmetric matrix so that

$$\hat{\mathbf{A}}_1 = \begin{bmatrix} 0 & -a_3 & a_2 \\ a_3 & 0 & -a_1 \\ -a_2 & a_1 & 0 \end{bmatrix} \tag{3}$$

where $a_i$ are the components of $\mathbf{A}_1$. At this point, one can argue that $\Theta(\mathbf{A}_1, \mathbf{e})$ hosts four degrees of freedom instead of three; however, one should also consider that the component of $\mathbf{e}$ parallel to $\mathbf{A}_1$ is obsolete. If $\mathbf{A}_1$ is bound to a space curve $s \to \mathbf{r}_c(s) \in \mathrm{R}^3$, forming a vector field $s \to \mathbf{A}_1(\mathbf{r}_c(s)) \in \mathrm{R}^3$ with a tangent relation as $\mathbf{A}_1(s) = \mathbf{r}_c'(s)$, where the apostrophe denotes $(\cdot)' = d(\cdot)/ds$ while $s$ is the arc-length, then one can find the curve's intrinsic frame (called the Frenet–Serret frame or TNB frame) by substituting $\mathbf{e}(s) = \mathbf{r}_c''(s)$ into Eq. (1). However, the Frenet–Serret frame is highly dependent on the space curve $\mathbf{r}_c(s)$ through differentiation, which makes principal cross-section directions hard to interpret, especially for