Physica A 482 (2017) 597–610

Contents lists available at ScienceDirect

Physica A

journal homepage: www.elsevier.com/locate/physa

Bifurcation and Turing patterns of reaction–diffusion activator–inhibitor model

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HIGHLIGHTS

- Hopf bifurcations of Gierer–Meinhardt model without and with diffusion are studied.
- Turing instability of such reaction–diffusion system is analyzed.
- Explicit formulas about the stability of the fixed point and limit cycle are derived.

ARTICLE INFO

Article history:
Received 6 December 2016
Received in revised form 12 February 2017
Available online 24 April 2017

Keywords:
Hopf bifurcation
Turing instability
Pattern
Activator–inhibitor model

ABSTRACT

Gierer–Meinhardt system is one of prototypical pattern formation models. Turing instability could induce various patterns in this system. Hopf bifurcation analysis and its direction are performed on such diffusive model in this paper, by employing normal form and center manifold reduction. The effects of diffusion on the stability of equilibrium point and the bifurcated limit cycle from Hopf bifurcation are investigated. It is found that under some conditions, diffusion-driven instability, i.e, Turing instability, about the equilibrium point and the bifurcated limit cycle will happen, which are stable without diffusion. Those diffusion-driven instabilities will lead to the occurrence of spatially nonhomogeneous solutions. As a result, some patterns, like stripe and spike solutions, will form. The explicit criteria about the stability and instability of the equilibrium point and the limit cycle in the system are derived, which could be readily applied. Further, numerical simulations are presented to illustrate theoretical analysis.

1. Introduction

Natural patterns are various in shape and form. The development processes of such patterns are complex, and also interesting to researchers. To understand the underlying mechanism for patterns of plants and animals, Alan Turing [1] first proposed the coupled reaction–diffusion equations. It was showed that the stable process could evolve into an instability with diffusion effect. He showed that diffusion could destabilize spatial homogeneous states and cause nonhomogeneous spatial patterns, which accounted for biological patterns in plants and animals. Such instability is frequently called Turing instability, also known as diffusion-driven instability. Gierer and Meinhardt [2] presented a prototypical model of coupled reaction–diffusion equations, which described the interaction between two substances, the activator and the inhibitor, and

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http://dx.doi.org/10.1016/j.physa.2017.04.053

was used to describe Turing instability. The Gierer–Meinhardt activator–inhibitor model is expressed in the following form

$$\begin{cases} \dfrac{\partial a}{\partial t} = \rho_0\rho + c\rho\dfrac{a^r}{h^s} - \mu a + D_a\dfrac{\partial^2 a}{\partial x^2}, \\ \dfrac{\partial h}{\partial t} = c'\rho'\dfrac{a^T}{h^u} - \upsilon h + D_h\dfrac{\partial^2 h}{\partial x^2} \end{cases} \tag{1}$$

where $a(x, t)$ and $h(x, t)$ represent the population density of the activator and inhibitor at time $t > 0$ and spatial location x, respectively. D_a and D_h are the diffusion constants of the activator and the inhibitor, respectively; $\rho\rho_0$ is the source concentration for the activator; ρ' is the one for the inhibitor; the activator and the inhibitor are removed by the first order kinetics μa and υh, respectively, either by enzyme degradation, or leakage, or re-uptake by the source, or by any combination of such mechanisms; now the sources of activator and the inhibitor are assumed to be uniformly distributed, that is, ρ and ρ' are constants.

Several results about such model have been achieved. If $s \neq u$, it is called to have different sources. If $s = u$, then it is called to be the model with common sources. For the model with common sources, Ruan [3] investigated the instability of equilibrium points and the periodic solutions under diffusive effects, which were stable without diffusion. There the perturbation method was employed to carry out the analysis. The criterion for Turing is dependent on the period of the limit cycle bifurcated from Hopf bifurcation. So it is not so easy to verify and apply. Refs. [4,5] also carried out the numerical analysis of the Turing instability for such model.

Liu et al. [6] also obtained the similar results about the same model, but with nonhomogeneous Dirichlet boundary conditions, using the bifurcation technique. By using the bifurcation technique, Liu et al. [6] investigated Hopf bifurcation and steady state bifurcation, and their interaction for this model and gave the conditions under which such bifurcation could happen. However, the model was subject to fixed Dirichlet boundary conditions. Recently, Song [7] further investigated the Turing–Hopf bifurcations and spatial resonance phenomenon. They found that spatial resonance could happen in the system, that is, there will be spatially resonant solutions. When $r = 2, s = 2, T = 1$ and $u = 0$, Wang et al. [8] studied its Turing instability and Hopf bifurcation. Also, Turing instability for the semi-discrete Gierer–Meinhardt model was considered in [9]. The influence of gene expression time delay on the patterns of Gierer–Meinhardt system was explored in Ref. [10]. Turing bifurcation in models like Brusselator and Gierer–Meinhardt systems were analyzed in Ref. [11]. Bifurcation for the activator–inhibitor model with saturation was analyzed in [12,13]. The existence and asymptotic behaviors of solutions and their stability in terms of diffusion effects have been extensively investigated, for example, [14–29] and references therein.

In application, we would like to know how the diffusion affects the stability of the bifurcated limit cycle and equilibrium points, and find the bifurcation direction, so that it could be verified if Turing instability will happen in the system. From above references, there are more results about bifurcation, such as Hopf bifurcation, Turing bifurcation and steady state bifurcation, while less results about the stability of bifurcated limit cycle under diffusion or the explicit and constructive justification formula for such Turing instability is absent. It is desirable to know the effects on the limit cycle and derive some stability criteria, which are explicit and constructive in form and easy to apply.

Now the case $u = 0, r = 2, T = 2$ and $s = 1$ will be considered in this paper, that is, the model is described as

$$\begin{cases} \dfrac{\partial a}{\partial t} = \rho_0\rho + c\rho\dfrac{a^2}{h} - \mu a + D_a\dfrac{\partial^2 a}{\partial x^2}, \\ \dfrac{\partial h}{\partial t} = c'\rho' a^2 - \upsilon h + D_h\dfrac{\partial^2 h}{\partial x^2}. \end{cases} \tag{2}$$

Through qualitative analysis, such as stability theory, normal form and bifurcation technique, we will investigated its Turing instability of the system with Neumann boundary conditions. It is found that under some conditions Turing instability will happen in the system. To be specific, the equilibrium point will become unstable from stable under diffusion, and the stable limit cycle from Hopf bifurcation will become unstable with diffusive effects. Consequently, some patterns appear. Spot and stripe patterns are identified numerically.

By using the scaling transformation, let $t = \frac{\tau}{\upsilon}, \bar{\mu} = \frac{\mu}{\upsilon}, D_H = \frac{D_h}{\upsilon}, D_A = \frac{D_a}{\upsilon}$.

$$\begin{cases} \dfrac{\partial A}{\partial \tau} = c_0 + \dfrac{A^2}{H} - \bar{\mu} A + D_A\dfrac{\partial^2 A}{\partial x^2}, \\ \dfrac{\partial H}{\partial \tau} = A^2 - H + D_H\dfrac{\partial^2 H}{\partial x^2}. \end{cases} \tag{3}$$

For simplicity, substitute $a, h, t, \mu, c, D_a, D_h$ for $A, H, \tau, \bar{\mu}, c_0, D_A, D_H$, respectively. System (3) can written as follows

$$\begin{cases} \dfrac{\partial a}{\partial t} = c + \dfrac{a^2}{h} - \mu a + D_a\dfrac{\partial^2 a}{\partial x^2}, \\ \dfrac{\partial h}{\partial t} = a^2 - h + D_h\dfrac{\partial^2 h}{\partial x^2} \end{cases} \tag{4}$$