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# A fractal growth model: Exploring the connection pattern of hubs in complex networks

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## HIGHLIGHTS

- A fractal growth model is proposed.
- Added edges inside boxes trigger the small-world-fractal transition and coexistence.
- Hub-nodes connection ratio $h_i$ was defined and analyzed.
- The distribution of $h_i$ show normal distribution in many real-world networks.

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## ABSTRACT

Fractal is ubiquitous in many real-world networks. Previous researches showed that the strong disassortativity between the hub-nodes on all length scales was the key principle that gave rise to the fractal architecture of networks. Although fractal property emerged in some models, there were few researches about the fractal growth model and quantitative analyses about the strength of the disassortativity for fractal model. In this paper, we proposed a novel inverse renormalization method, named Box-based Preferential Attachment (BPA), to build the fractal growth models in which the Preferential Attachment was performed at box level. The proposed models provided a new framework that demonstrated small-world-fractal transition. Also, we firstly demonstrated the statistical characteristic of connection patterns of the hubs in fractal networks. The experimental results showed that, given proper growing scale and added edges, the proposed models could clearly show pure small-world or pure fractal or both of them. It also showed that the hub connection ratio showed normal distribution in many real-world networks. At last, the comparisons of connection pattern between the proposed models and the biological and technical networks were performed. The results gave useful reference for exploring the growth principle and for modeling the connection patterns for real-world networks.



## 1. Introduction

Network theory has proven a powerful framework for studying the robustness [1–8], spread [9,10], synchronization and control [11–15] on complex systems that were retrieved from many real systems such as neural system, brain, biological system, scientist collaboration relationship. Hereinto, the topology of the generated complex network was the fundamental

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factor to the above researches [10,11]. Small-world, scale-free, fractal and community were commonly found in many real-world networks [16–21].

Self-similar fractal property can be described when boxes are used to cover many real networks. The minimum number of boxes and the size of the boxes demonstrate the power-law relationship [18]. Fractal network can be characterized by the two scaling relations [18,20,21]:

$$N_B\left(l_B\right)/N_0 \sim l_B^{-d_B} \tag{1}$$

$$k_B(l_B)/k_{hub} \sim l_B^{-d_k} \tag{2}$$

where $N_B(l_B)$ is the minimum number of boxes when covering the network with the boxes whose size is $l_B$. $d_B$ is the fractal dimension. $k_{hub}$ and $k_B(l_B)$ are the degree of the most connected node inside each box and that of each box [18,21], respectively. $d_k$ means the growth dimension of the degree. Numerous mechanisms have been proposed to simulate the properties of complex network. Barabási and Albert [17] were the first who introduce the Preferential Attachment principle to model the growth scale free networks, and then followed by many models [22–33]. Among them, the skeleton [30] and the strong disassortativity [20,21] between the most connected nodes (hubs) were two mainstream to explain the origin of self-similarity in growth network. Disassortativity means that the hubs prefer to grow by connections to less-connected nodes rather than to other hub. Recently, a structural preferential attachment (SPA) method [27,28] was proposed to model the emergence of the universal properties of complex networks. The generated model showed that the universal properties of complex networks were the consequence of preferential attachment at the higher level.

Meanwhile, there was a conundrum about small-world and fractal features in many real-world networks. Self-similar scale-free feature indicates a power-law dependence of the distance with the network size [20,21,34], $D \sim N_0^{1/d_B}$, where $D$ is the diameter of the network and $D = l_B$. Generally, small-world property is expressed by the logarithmic increase of $D$ with the total number of the nodes $N_0$ [35]: $D \sim \ln N_0$. Paradoxically, the diameter of the network showed linear growth in small-world network and exponential growth in self-similar network. Actually, fractal and small-world properties coexist in many real networks, such as cellular network protein–protein interaction network, brain networks, the WWW, actor network, and so on. So, it is necessary to build a framework that reconciles the contradictory aspect, fractal and the small-world property. *Gallos* et al. incorporated weaker ties to the network and converted the large-world self-similar into a small-world network, which showed the "the strength of weak ties" in brain networks [36]. Similarly, Rozenfeld et al. provided a framework based on renormalization group (GR) theory to identify the small-world-fractal transition [34]. They started with a fractal network and added shortcuts between two nodes according to the probability $p(r) = Ar^{-\alpha}$, where $r$ was the distance between the nodes. According to $\alpha$, after the renormalization, the network topologies were classified into three universality classes: pure small-world, fractal with shortcuts and fixed point. Meanwhile, *Kawasaki* et al. [37] showed that the fractal and small-world properties can crossover from one to the other by varying unique characteristic length scale. The scale analyses were based on many real-world complex networks.

As mentioned above, the fractal models were built according to a set of fixed rules and did not involved any stochastic connection. All of the frameworks started with a large-world fractal network model which was created by a set of fixed rules [18,20,34], and changed the connection of the edges or added redundant edges to the networks. So far, it is still not clear about the small-world-fractal transition in the growth network model, and it is not clear about the effect of the structure in the modules on the global features of the network.

Here, we constructed the fractal growth models through performing the Preferential Attachment at box level in which stochastic choice was introduced. Based on the inverse renormalization and Preferential Attachment, we proposed the BPA method to model the fractal growth network. And based on the BPA growth models, a new framework was built to demonstrate the small-world-fractal transition and coexistence. Finally, the comparisons of the connection pattern between the BPA models and the real-world networks gave useful guidelines about the growth way of these real-world fractal networks.

## 2. BPA models

### 2.1. BPA model building

In the process of renormalization, each box was replaced by a node. These renormalized nodes were connected if there was at least one link between the corresponding boxes. On the contrary, the inverse renormalization expanded a node into a box. In consideration of Eqs. (1) and (2), the increase scale of the box was limited by the growth multiple of node $n$. Given box size $l_B$, it ought to meet the multiplicative growth rules: $N_1 = nN_0$, $k_{Bi} = sk_i$, where $N_0$ was the number of node in origin network; $N_1$ was the number of node after an inverse renormalization; $k_i$ was the degree of the node $i$ in origin network, and $k_{Bi}$ was the degree of the hub node in box $i$ after an inverse renormalization. $n > 1$ and $s > 1$ are time-independent constants. Here, $n$ determines the density of nodes in the new network and $s$ means the incremental scale of the degree. Here we defined two probabilities to describe the connection status of box $i$: the forming probability $p_i$ and $c$ as shown in Eqs. (3) and (4).

$$p_i = \frac{N_{Bi}}{1 + (s-1)k_{Bi}r_B} \tag{3}$$