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Dynamic optimization and its relation to classical and quantum constrained systems

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HIGHLIGHTS

- Dynamic optimization problem.
- Constrained systems and its applications.
- Dirac's method in finance.
- Quantum mechanics and its applications.

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ABSTRACT

We study the structure of a simple dynamic optimization problem consisting of one state and one control variable, from a physicist's point of view. By using an analogy to a physical model, we study this system in the classical and quantum frameworks. Classically, the dynamic optimization problem is equivalent to a classical mechanics constrained system, so we must use the Dirac method to analyze it in a correct way. We find that there are two second-class constraints in the model: one fix the momenta associated with the control variables, and the other is a reminder of the optimal control law. The dynamic evolution of this constrained system is given by the Dirac's bracket of the canonical variables with the Hamiltonian. This dynamic results to be identical to the unconstrained one given by the Pontryagin equations, which are the correct classical equations of motion for our physical optimization problem. In the same Pontryagin scheme, by imposing a closed-loop λ-strategy, the optimality condition for the action gives a consistency relation, which is associated to the Hamilton–Jacobi–Bellman equation of the dynamic programming method. A similar result is achieved by quantizing the classical model. By setting the wave function $\Psi(x,t) = e^{iS(x,t)}$ in the quantum Schrödinger equation, a non-linear partial equation is obtained for the S function. For the right-hand side quantization, this is the Hamilton–Jacobi–Bellman equation, when $S(x,t)$ is identified with the optimal value function. Thus, the Hamilton–Jacobi–Bellman equation in Bellman's maximum principle, can be interpreted as the quantum approach of the optimization problem.

1. Introduction

We have recently witnessed the increasing application of ideas from physics to finance and economics, such as path integral techniques applied to the study the Black–Scholes model in its different forms [1–7]. Some developments have also been used to try to understand the Black–Scholes equation as a quantum mechanical Schrödinger equation [8–10]. In the

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last few years, constrained systems techniques, through Dirac's method [11,12] have been used to explain some features of stochastic volatility models [13,14] and the multi-asset Black–Scholes equation [15]. In this paper, we apply these same constrained methods to understand (from a physical point of view) a particular issue: the dynamic optimization problem.

We start by analyzing the dynamic optimization problem for a single-state variable x and a control variable u. By identifying the state variable x as the coordinate of a physical particle and the Lagrange multiplier λ as its canonical momentum p_x, we can map the theory in the Hamiltonian phase space. Here, the model presents constraints thus, it is necessary (to study the system in a correct way) to use Dirac's method of constrained systems. The application of this method implies that constraints are of second-class character according to Dirac's classification. Thus, the dynamic optimization problem can be seen as a second-class physically constrained system.

We also analyze the role of open-loop and closed-loop strategies in the context of Pontryagin's framework. We explicitly show that the only consistent strategies that permit the Pontryagin equations to be obtained correctly from the optimization of cost functionals are open-loop λ-strategies ($\lambda = \lambda(t)$). For closed loop λ-strategies ($\lambda = \lambda(x, t)$), the optimization of the cost functional gives a consistency relation which is related to the Hamilton–Jacobi–Bellman equation.

After that, we explore the quantum side of this classically constrained system. By quantizing it according to the standard rules of quantum mechanics and imposing the constraints as operator equations over the wave function, we arrive at a set of partial differential equations for the wave function. After defining the wave function as $\Psi(x, t) = e^{iS(x,t)}$, these equations map into some partial differential equations for the S function. For right-hand side quantization order, these equations give origin to the Hamilton–Jacobi–Bellman equation of the dynamic programming theory. Thus, Bellman's maximum principle can be considered as the quantum view of the optimization problem.

To make this paper self-contained for non-physicist readers coming from the optimization field, we start with a brief digression on classical and quantum physics in Section 2.

2. Quantum and classical mechanics

2.1. Hamiltonian quantum and classical mechanics

In physics, quantum-dynamic behavior is defined by the Hamiltonian operator. For the simple case of a nonrelativistic one-dimensional particle subjected to external potential $U(x)$, the Hamiltonian operator reads

$$\check{H} = \frac{1}{2m}\check{P}_x^2 + U(x) = -\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2} + U(x),$$

where $\check{P}_x = -i\hbar\frac{\partial}{\partial x}$ is the momentum operator. The wave function at time t (given that the wave function at $t = 0$ is Ψ_0) is thus

$$\Psi(x, t) = e^{-\frac{i}{\hbar}\check{H}t}\Psi_0(x),$$

which can be written as a convolution according to

$$\Psi(x, t) = \int K(x, t|x'0)\Psi_0(x')dx',$$

where the propagator K admits the Hamiltonian Feynman path integral representation:

$$K(x, t|x'0) = \int \mathcal{D}x\frac{\mathcal{D}p_x}{2\pi}\exp\left(\frac{i}{\hbar}A[x, p_x]\right). \tag{1}$$

The symbol $\int \mathcal{D}x\frac{\mathcal{D}p_x}{2\pi}$ denotes the sum over the set of all trajectories that start at (x_0, p_x^0) for $t = 0$ and end at (x, p_x) at time t in the phase space. Thus, all trajectories are needed to evaluate the quantum propagator. In (1), $A[x, p_x]$ is the classical Hamiltonian action functional, given by

$$A[x, p_x] = \int p_x\dot{x} - H(x, p_x)\,dt,$$

where H is the Hamiltonian function

$$H(x, p_x) = \frac{1}{2m}p_x^2 + U(x). \tag{2}$$

From the classical dynamics (Newtonian equations) of the quantum system, we understand the particular trajectory in the phase space (x, p_x) that gives an extreme to the Hamiltonian functional $A[x, p_x]$: the path for which the variation of the action is zero:

$$\delta A[x, p_x] = A[x + \delta x, p_x + \delta p_x] - A[x, p_x] = 0.$$

We can show that this classical trajectory necessarily satisfies the Hamiltonian equations of motion:

$$\dot{x} = \frac{\partial H(x, p_x)}{\partial p_x}, \qquad \dot{p}_x = -\frac{\partial H(x, p_x)}{\partial x}. \tag{3}$$