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Oligarchy as a phase transition: The effect of wealth-attained advantage in a Fokker–Planck description of asset exchange ☆

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A B S T R A C T

The "Yard-Sale Model" of asset exchange is known to result in complete inequality—all of the wealth in the hands of a single agent. It is also known that, when this model is modified by introducing a simple model of redistribution based on the Ornstein–Uhlenbeck process, it admits a steady state exhibiting some features similar to the celebrated Pareto Law of wealth distribution. In the present work, we analyze the form of this steady-state distribution in much greater detail, using a combination of analytic and numerical techniques. We find that, while Pareto's Law is approximately valid for low redistribution, it gives way to something more similar to Gibrat's Law when redistribution is higher. Additionally, we prove in this work that, while this Pareto or Gibrat behavior may persist over many orders of magnitude, it ultimately gives way to gaussian decay at extremely large wealth. Also in this work, we introduce a bias in favor of the wealthier agent – what we call Wealth-Attained Advantage (WAA) – and show that this leads to the phenomenon of "wealth condensation" when the bias exceeds a certain critical value. In the wealth-condensed state, a finite fraction of the total wealth of the population "condenses" to the wealthiest agent. We examine this phenomenon in some detail, and derive the corresponding modification to the Fokker–Planck equation. We observe a second-order phase transition to a state of coexistence between an oligarch and a distribution of non-oligarchs. Finally, by studying the asymptotic behavior of the distribution in some detail, we show that the onset of wealth condensation has an abrupt reciprocal effect on the character of the non-oligarchical part of the distribution. Specifically, we show that the above-mentioned gaussian decay at extremely large wealth is valid both above and below criticality, but degenerates to exponential decay precisely at criticality.

1. Introduction

1.1. Motivation and prior work

Asset Exchange Models (AEMs) were first proposed by Angle [1] in 1986 in the social sciences literature. AEMs are collections of N economic agents, each of which possesses some amount of wealth, and engages in pairwise transactions according to certain idealized rules. These interactions are usually designed to conserve both the total number of agents

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N and the total wealth W for a closed economy, though it is also certainly possible to extend the model to account for production, consumption, immigration and emigration. In the continuum limit, the wealth distribution can be described by an agent density function $P(w, t)$, the first two moments of which are the total number of agents $N = \int_0^\infty dw\ P(w, t)$ and the total wealth $W = \int_0^\infty dw\ P(w, t)w$.

In the late 1990s, several seminal papers appeared that transformed "econophysics" into a rigorous and quantitative science. Ispolatov, Krapivsky and Redner [2] showed how to derive a Boltzmann equation for the time evolution of $P(w, t)$ for a particular AEM, in which the losing agent is selected with even odds and the amount lost is a fraction of the wealth of the losing agent. So far as we are aware, this was the first application of the methods of non-equilibrium kinetic theory to AEMs.[1] Meanwhile, Drăgulescu and Yakovenko published the treatise "Statistical mechanics of money" [4], in which they clearly demonstrated that wealth distributions arise naturally and endogenously, even among identical traders, just as velocity distributions arise among identical molecules in statistical mechanics. The latter point in particular was, and is, controversial for economists, who would prefer to believe that identical traders experience identical outcomes, so that any distribution in their outcomes must be due to their individual qualities. Finally, Chakraborti and Chakrabarti [5] presented a detailed analysis of a number of statistical mechanical models of money, pointing out the important effect of saving propensity on its distribution.

A more economically realistic AEM was then proposed by Chakraborti [6] in 2002, and lucidly described by Hayes [7] shortly afterward who labeled it the "Yard Sale Model" (YSM). In this model, the losing agent is selected with even odds, but the amount lost is a fraction β of the wealth of the *poorer* agent. This assumption is perhaps more realistic than that in [2] because most economic agents engage in transactions for which the amount at stake is strictly less than their own total wealth. Chakraborti presented numerical evidence that this model results in the concentration of all wealth in the hands of a single agent, in spite of the fact that the losing agent is selected with even odds. This latter phenomenon, called "wealth condensation," was first reported by Bouchaud and Mézard in 2000 and later studied by Burda et al. [8]. More generally, it is characterized by the concentration of a finite fraction of wealth in the hands of a single agent, and it is thought to provide a statistical mechanical explanation of the phenomenon of oligarchy.

In 2007, Moukarzel et al. [9] modified the YSM by biasing the selection of the losing agent so as to confer a fixed advantage to the poorer of the two transacting agents. He then analyzed the steady-state master equation for this biased YSM, and demonstrated that it exhibits a first-order phase transition from the fully *wealth-condensed* state that Chakraborti observed to a stable distribution. In other words, if the introduced bias favoring the poorer of the two agents was sufficiently large, it could inhibit wealth condensation.

In 2014 Boghosian [10] derived the Boltzmann equation for the YSM, and showed that it reduces to a nonlinear, integrodifferential Fokker–Planck equation in the limit of small β, which he termed the *small-transaction limit*. The derivation is similar to that of the nonlinear, integrodifferential Fokker–Planck collision operator used in plasma physics from the Boltzmann equation in the weak-collision limit [11]. This Fokker–Planck description is significant for its *universality*: Though different posited distributions for β would result in different Boltzmann equations, the Fokker–Planck equation obtained from all of them in the small-transaction limit is universal in form.

Here another analogy may be made with kinetic theory: While different dilute gases may exhibit very different collision dynamics at the molecular level, and therefore be described by very different Boltzmann equations, the Chapman–Enskog analysis teaches us that they all reduce to the Navier–Stokes equations in the limit of small Knudsen number. The universality of the Fokker–Planck description for the macroscopic description of wealth distributions [10] is likewise analogous to that of the Navier–Stokes equation for the macroscopic description of dilute gases, and the small transaction limit plays the role of small Knudsen number in this metaphor.

Later in 2014, Boghosian [12] showed that this universal Fokker–Planck equation could be derived directly from the underlying stochastic process, without the intermediary of the Boltzmann equation. As noted above, a number of prior numerical studies [6,10,12] had presented numerical evidence that wealth concentrates without bound in the YSM, unless it is supplemented with some model for redistribution. This latter point was then proven rigorously by Boghosian, Johnson and Marcq in 2015 [13], who showed that the Gini coefficient is a Lyapunov functional of both the Boltzmann and the universal Fokker–Planck equations for the model. This work also demonstrated that the time-asymptotic value of the Gini coefficient of the non-redistributive model is unity, corresponding to absolute oligarchy.

At first glance, the instability of the YSM without redistribution seems counter-intuitive since the losing agent in any transaction is selected with even odds. The fact that the amount at stake in any transaction is always a smaller fraction of the wealth of the poorer agent than of the richer agent, however, means that the latter is able to withstand a longer string of losses, and it is this that ultimately breaks the symmetry. It may be noted in passing that this is consistent with Keynesian economic theories which suggest that market economies are inherently unstable without some kind of government intervention—e.g., redistribution.

When the YSM is stabilized by a model of redistribution, based on the Ornstein–Uhlenbeck process [14], numerical evidence has been presented [10,12] indicating that the resulting model yields a steady-state that shares certain features

[1] For a general review of modern applications of kinetic theory that puts this important work in some context, the textbook by Krapivsky, Redner and Ben Naim is recommended [3].