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Fractality and scale-free effect of a class of self-similar networks

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HIGHLIGHTS

- We construct a class of self-similar growing networks from a directed graph.
- Our networks have fractality.
- Our networks have scale-free effect.

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ABSTRACT

In this paper, given an initial directed graph as a self-similar pattern and fix two nodes in the pattern, we can iterate the graph by replacing any directed edge with the initial graph of pattern and identifying the fixed nodes of pattern with the endpoints of directed edge. Using the iteration again and again, we obtain a family of growing self-similar networks. Modify these networks to be undirected ones, we obtain growing self-similar undirected networks. We obtain the fractality of our self-similar networks and find out the scale-free effect in terms of the matrix related to two fixed nodes in the initial graph.

1. Introduction

The study of complex networks is a young and active area of scientific research inspired by the study of real-world networks such as computer networks, brain networks and social networks. Please refer to [1–4] for complex networks.

Self-similarity of fractal introduced by Mandelbrot [5], is one of the most influential results of contemporary mathematics. How to apply self-similarity and fractality to complex networks? Song et al. [6–8] and Gallos et al. [9] studied this topic. According to [6–9], a network is *(statistically) self-similar* if the power exponent of degree distribution is invariant under renormalization, an l-box is a subset of node set V such that the shortest distance between any two nodes in the subset is less than l, and fractal dimension d_B is given by $\#V/N(l) \sim l^{d_B}$, where $N(l)$ is the smallest number of l-boxes needed to cover the network. Goh, Slavi, Kim and Kahng also presented some approaches to analyze networks that reveals the underlying self-similarity [10].

Through the iteration function system (IFS), Hutchinson [11] defined rigorously the self-similar fractal structure, which is a *deterministic self-similarity* far from the random self-similarity in real networks as above. It is worth noting that the IFS

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Fig. 1. Model 1: change an edge into a symmetrical graph.

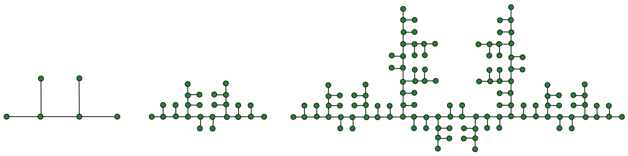

Fig. 2. $\overline{G}_1, \overline{G}_2, \overline{G}_3$ of growing networks w.r.t. model 1.

is an analogy of symbolic system $\Sigma_n = \{x_1x_2\cdots x_t\cdots : x_i \in [1, n]\cap\mathbb{N}\}$. Then Komjáhy and Simon [12] constructed a family of growing networks $\{G_t\}_t$ such that G_t has node set $V_t = \{x_1x_2\cdots x_t : x_i \in [1, n]\cap\mathbb{N}\}$, and obtain an interesting result that for any $x \in [1, n]\cap\mathbb{N}$ and $a, b \in V_{t-1}$, if we denote by d_i the shortest distance on G_i, then $\mathrm{d}_t(xa, xb) = \mathrm{d}_{t-1}(a, b)$, this formula is somewhat of self-similarity in Hutchinson's sense.

Compared to the *algebraic* approach in [12], in this paper we return to the *geometric and deterministic self-similar networks* by replacing an edge with an initial graph. For example shown in Figs. 1 and 2, given a model changing an edge into a symmetrical and undirected initial graph G, there exists a family of iterating and growing networks $\overline{G}_1(= G), \overline{G}_2, \overline{G}_3, \ldots$ naturally.

However, when the self-similar pattern G is *unsymmetrical*, the above growing process is undeterministic. For this, we will introduce the *directed* graph model and its growing networks as follows.

Fix a directed graph $G = (V, E)$ and distinct nodes $A, B \in V$ such that A, B are not neighbors, we can generate a sequence $\{\overline{G}_t\}$ of growing undirected networks.

We will describe the iterating process by induction. For $t = 1$, we let $G_1 = G$ and $\overline{G}_1$ the modified undirected graph of G_1. We always assume that $\overline{G}_1$ is connected. Assume that $G_t = (V_t, E_t)$ with $A, B \in V_t \cap V$, which implies that we always preserve the nodes A and B with their notation during the inductive process. We can also think A and B as the starting node and ending node of V_t formally, i.e., there is a formal (but not real) direction from A to B in G_t.

For time $(t + 1)$, we obtain $\#E$ copies $\{G_t \times e\}_{e\in E}$ of G_t.

Step 1. Replacement: Focusing on the initial graph G, for every edge e in G, we replace e by one copy $G_t \times e$ with node set $V_t \times e$. In fact, we delete all edges in G and keep all nodes of G in this step.

Step 2. Identification: Identify $(A, e)(\in V_t \times e)$ with the starting node of e, and identify $(B, e)(\in V_t \times e)$ with the ending node of e, where the identification is denoted by $\simeq$.

Then as in Fig. 3, we obtain a new directed network G_{t+1} by replacing each edge of G by one copy of G_t and identifying the starting (or ending) nodes of the edge and the corresponding copy of V_t, that means the direction of the copy of V_t coincides with that of the edge. Keep the nodes A, B together with their notation for time $(t + 1)$.

Step 3. Modification: Modifying these directed networks to be undirected ones, we obtain self-similar networks $\{\overline{G}_t\}$.

We will provide an example to illustrate our recursive construction in Fig. 4.

Example 1. Suppose the initial direct graph $G = G_1$ consists of six nodes and six edges, where A is the starting node and B is the ending node, although there is no directed path from A to B. In Step 1 we color the starting node red and the ending node green, and ignore the nodes of G. In step 3 we give a undirected graph $\overline{G}_2$.

Now our growing self-similar networks $\{\overline{G}_t\}_t$ are constructed, and thus we will investigate the fractality and power law of cumulative degree distribution.

We introduce the notation before the statements of our main results. For the initial self-similar pattern G, let $d(\geq 2)$ denote the shortest distance between A and B in $\overline{G}_1$, and $m = \#E$ the cardinality of directed edges in G. Write $\gamma = \max_{z\in G} \deg_G(z)$. Denote by x_{out} (outdegree) and x_{in} (indegree) the number of edges starting and ending at x respectively. Then

$$M = \begin{pmatrix} A_{out} & A_{in} \\ B_{out} & B_{in} \end{pmatrix}$$

is a non-negative matrix where $\deg(A), \deg(B) \geq 1$ where $\deg(x) = x_{out} + x_{in}$. Suppose $\lambda \geq 1$ is the maximal (Perron–Frobenius) eigenvalue of M.

Theorem 1. *The self-similar networks $\{\overline{G}_t\}_t$ satisfy the fractality, i.e.,*

$$\frac{\#V_t}{N_t(l)} \sim l^{\frac{\log m}{\log d}},$$

where $N_t(l)$ is the smallest number of l-boxes needed to cover V_t.