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Spectral coarse grained controllability of complex networks

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HIGHLIGHTS

- Spectral coarse grained controllability (SCGC) of complex networks are investigated.
- The ER, SF and SW random networks are with distinct SCGC properties.
- The SW networks are very robust, while the SF networks are sensitive to the SCGC.
- The robust SCGC of random ER networks is an emergent property.
- Some social and biological networks are hard to be controlled during the SCGC.

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ABSTRACT

With the accumulation of interaction data from various systems, a fundamental question in network science is how to reduce the sizes while keeping certain properties of complex networks. Combined the spectral coarse graining theory and the structural controllability of complex networks, we explore the structural controllability of undirected complex networks during coarse graining processes. We evidence that the spectral coarse grained controllability (SCGC) properties for the Erdös–Rényi (ER) random networks, the scale-free (SF) random networks and the small-world (SW) random networks are distinct from each other. The SW networks are very robust, while the SF networks are sensitive during the coarse graining processes. As an emergent properties for the dense ER networks, during the coarse graining processes, there exists a threshold value of the coarse grained sizes, which separates the controllability of the reduced networks into robust and sensitive to coarse graining. Investigations on some real-world complex networks indicate that the SCGC properties are varied among different categories and different kinds of networks, some highly organized social or biological networks are more difficult to be controlled, while many man-made power networks and infrastructure networks can keep the controllability properties during the coarse graining processes. Furthermore, we speculate that the SCGC properties of complex networks may depend on their degree distributions. The associated investigations have potential implications in the control of large-scale complex networks, as well as in the understanding of the organization of complex networks.

1. Introduction

Controllability of complex networks is in the focus of complex network science and control science in recent years [1–10]. Structural controllability is a typical topic. The structural controllability framework [11,12] assumes nodes in the network

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follow the canonical linear, time-invariant dynamics

$$\frac{d\mathbf{x}}{dt} = \mathbf{A}\mathbf{x}(t) + \mathbf{B}\mathbf{u}(t), \tag{1}$$

where $\mathbf{x}(t) = (x_1(t), \ldots, x_N(t))^T$ describes the state of N nodes at time t in the system. $\mathbf{A}$ is a $N \times N$ matrix, which captures the topological structure of the network. The elements of $\mathbf{A}$ are nonnegative, which describe the wiring strengths between components. $\mathbf{B}$ is a $N \times M$ input matrix ($M \leq N$), which describes the nodes controlled by outside controllers. $\mathbf{u}(t) = (u_1(t), \ldots, u_M(t))^T$ is the input vector. For the structural controllability of system (1), it is reported that the following statements are equivalent [13,14]:

(i) System (1) is completely controllable.
(ii) Rank condition: The controllability matrix

$$\mathbf{C} = [\mathbf{B}, \mathbf{AB}, \mathbf{A}^2\mathbf{B}, \ldots, \mathbf{A}^{N-1}\mathbf{B}] \tag{2}$$

has full rank, that is $rank(\mathbf{C}) = N$.
(iii) Popov–Belevitch–Hautus (PBH) eigenvector test: The relationship $\mathbf{v}^T\mathbf{A} = \lambda\mathbf{v}^T$ implies $\mathbf{v}^T\mathbf{B} \neq 0^T$, where $\mathbf{v}$ is a nonzero left eigenvector of $\mathbf{A}$ corresponding to the eigenvalue λ.

For large networks, it is computationally difficult to verify the rank condition (ii) and the PBH eigenvector test (iii). In 2011, based on the rank condition, Liu et al. [1] developed a fast tool to investigate the structural controllability of large-scale directed and weighted complex networks, and they solved the minimum input problem. In 2013, based on the PBH eigenvector test, Yuan et al. [9] developed an exact controllability paradigm to achieve full control of networks with arbitrary structures and link-weight distributions, which solved the structural controllability of undirected complex networks. From Yuan et al. [9], the minimum number of controllers N_D is determined by the maximum geometric multiplicity $\mu(\lambda_i)$ of the eigenvalue λ_i for the adjacency matrix $\mathbf{A}$:

$$N_D = \max_i\{\mu(\lambda_i)\}. \tag{3}$$

For large sparse networks with a small fraction of self-loops, N_D is determined by the rank of $\mathbf{A}$:

$$N_D = \max\{1, N - rank(\mathbf{A})\}. \tag{4}$$

For large dense networks with identical link weights w and a small fraction of self-loops, N_D can be determined by:

$$N_D = \max\{1, N - rank(wI_N + \mathbf{A})\}, \tag{5}$$

where $\mathbf{I}_N$ is the $N \times N$ identity matrix. It is noted that, many real-world complex networks are sparse, especially those arising from biological systems and other real-world systems [15,16].

One of the biggest challenges in complex network science is the sheer size of large complex networks. Many mathematical algorithms used to investigate complex networks run in times that grow polynomially with the sizes of the networks, therefore, quantitative evaluation of large complex networks is a complicated and often difficult task [17]. The recently proposed spectral coarse graining scheme is a promising way to reduce the complexity of large networks. The coarse graining scheme maps the original network into a smaller one, while the spectral and synchronization properties of the initial system are preserved in the reduced networks [17–19].

A natural question is whether the controllability property can be preserved during the spectral coarse graining processes. In other words, if the initial network is coarse grained into small ones, can the ratio of minimum driver nodes $n_D = N_D/N$ be preserved? In the following, we call the above mentioned question as spectral coarse grained controllability (SCGC). It is interesting to explore the SCGC properties for different types of random complex networks and real-world ones. Motivated by these issues, we give a brief overview on the spectral coarse graining process in Section 2, and investigate the SCGC of three classes of random undirected complex networks in Section 3. In Section 4, the SCGC of some real-world undirected complex networks are investigated. Possible physical mechanisms explanations will be presented in Section 5. Discussions and some concluding remarks are in the last Section 6.

2. Spectral coarse graining of complex networks

The spectral coarse graining strategy is based on the idea of grouping nodes with similar spectral components together. The aim is to obtain a reduced network that preserves some properties of the initial network. The properties of interest can be the main characteristics of random walks [17], synchronization [18,19] and so on.

Suppose $\mathbf{A} = (a_{ij})_{N\times N}$ is the adjacency matrix of a complex network, then stochastic matrix $\mathbf{W} = (w_{ij})_{N\times N}$ with

$$w_{ij} = \frac{a_{ij}}{\sum_{k=1}^{N} a_{ik}}$$