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Volatility smile as relativistic effect

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HIGHLIGHTS

- We give a probability distribution for a relativistic extension of Brownian motion.
- It is (1) is properly normalized and (2) obeys the tower law (semigroup property).
- The model is a 1-constant-parameter extension of the Black–Scholes–Merton model.
- Relativistic effects yield fat tails and, thus, volatility smiles in options prices.
- Quantum field theory should provide local description of such nonlocal processes.

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ABSTRACT

We give an explicit formula for the probability distribution based on a relativistic extension of Brownian motion. The distribution (1) is properly normalized and (2) obeys the tower law (semigroup property), so we can construct martingales and self-financing hedging strategies and price claims (options). This model is a 1-constant-parameter extension of the Black–Scholes–Merton model. The new parameter is the analog of the speed of light in Special Relativity. However, in the financial context there is no "speed limit" and the new parameter has the meaning of a characteristic diffusion speed at which relativistic effects become important and lead to a much softer asymptotic behavior, i.e., fat tails, giving rise to volatility smiles. We argue that a nonlocal stochastic description of such (Lévy) processes is inadequate and discuss a local description from physics. The presentation is intended to be pedagogical.

1. Introduction

A host of mathematical methods for describing physical phenomena have been gobbled up into quantitative finance. One of the most vivid examples is Brownian motion. Methods of classical mechanics, statistical physics, and even quantum mechanics (see below), are widely used in quantitative finance. One area of physics that – to a large extent – has eluded financial applications is Einstein's Special Relativity.

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There are two reasons. First, in most aspects of finance (e.g., asset pricing) the finiteness of the speed of light is utterly irrelevant, so any direct application of Special Relativity would be farfetched at best.[3] Second, in Special Relativity spacetime has a Minkowski signature, e.g., in 2-dimensional Minkowski spacetime (with one spatial x and one time t direction) the distance between two points is given by $s^2 = (x_1 - x_2)^2 - (t_1 - t_2)^2$. However, Minkowski spacetime is utterly inapplicable in finance, at least in the context of asset pricing. Thus, if we take Brownian motion, it is equivalent to a free quantum mechanical particle in Euclidean time, not in usual (Minkowski) time. Euclidean time t_E is related to Minkowski time t_M via the so-called Wick rotation $t_M = -it_E$ [7], i.e., in finance we are in imaginary time as far as physics is concerned. Therefore, 2-dimensional spacetime that might be relevant in the financial context would have to have a Euclidean signature with the distance given by $s^2 = (x_1 - x_2)^2 + (t_1 - t_2)^2$. And this makes all the difference.

For relativistic effects to be relevant in finance, there is no need for them to be literally related to (the finiteness of) the speed of light—the latter pertains to Minkowski spacetime and is not of much relevance in finance (cf. fn. 3). Instead, relativistic effects in Euclidean spacetime have nothing to do with the speed of light (as a constant of nature), but, e.g., with a characteristic speed of diffusion (in the Brownian motion lingo) at which higher-order (in powers of the diffusion speed) effects can no longer be neglected. A priori these higher-order effects can be whatever we wish them to be. However, we can constrain them by requiring that they are of the same form as analogous effects in Special Relativity in Minkowski spacetime. Practically speaking, we can start with a relativistic theory in Minkowski time and Wick-rotate it to Euclidean time. As we will see below, this almost gives us the right answer, but not quite. And the right answer is rather intriguing. . . .

Our motivation and context for doing all this are volatility smiles in option prices. They are artifacts of using Gaussian probability distributions as in the Black–Scholes–Merton (BSM) model [8,9]. Real-life distributions are fat-tailed and result in volatility smiles (see, e.g., [10]). There are various extensions of the BSM model such as stochastic volatility[4] and local volatility[5] models, which lead to fat tails. As it turns out, relativistic effects also give rise to fat tails. The aforesaid higher-order effects soften the decay of the probability distribution at large x, hence fat tails. So, the idea is to make Brownian motion relativistic in Euclidean (not Minkowski) time. The analog of the speed of light then is a characteristic diffusion speed at which relativistic effects are sizable.

This is precisely the approach we follow here. Most of the literature on relativistic Brownian motion has been focused on Minkowski spacetime.[6] Even the telegraph equation, which arises in relativistic heat conduction,[7] is in Minkowski time and is not applicable for our purposes here.[8] So, we discuss how to construct the probability distribution for relativistic Brownian motion in Euclidean time. A natural way to proceed is to start from the action for a 1-dimensional relativistic point particle in Minkowski time and Wick-rotate it to Euclidean time. The probability distribution then can be obtained using path integral techniques by writing the action in a reparametrization invariant form, fixing the gauge and reducing the path integral to a Gaussian (i.e., calculable) form,[9] which can be readily evaluated. This would likely be a high energy theorist's approach. However, if we (naturally) assume the path integral measure to be the simplest possible one, then we run into two problems. First, while the resultant probability distribution is expressly invariant under $SO(2)$ rotations in 2-dimensional Euclidean spacetime, it does not normalize to 1 when we integrate over all positions x. Instead, the normalization decays exponentially in time and – in the financial context – corresponds to a fixed constant interest rate. Its physical interpretation is Einstein's famous rest energy $E_0 = mc^2$, where m is the mass of the particle and c is the speed of light.[10] We must remove this contribution at the expense of giving up the aforesaid $SO(2)$ rotational invariance, which is just as well as it is clearly absent in the context of finance. Second, even ignoring the rest energy issue, the probability distribution does not obey the tower law (a.k.a. semigroup property), which is a serious issue as it is utterly unclear how to construct martingales and price options and other claims in the absence thereof (see below).

We should emphasize that it is not that the aforesaid path integral computation is wrong. It is just that the naïve (i.e., the simplest) choice of the path integral measure is not suitable for our intended financial applications. Happily, there is a rather simple way to circumvent this issue. We can use the Hamiltonian approach, where the probability distribution obeys a first-order (in time derivatives) differential equation. The price to pay is that in the relativistic case the Hamiltonian is a non-polynomial (i.e., nonlocal) differential operator w.r.t. x derivatives. However, a priori this does not pose an issue and,

3 Albeit, it has in fact been argued in the context of high frequency trading (HFT) that relativistic effects literally may become relevant therefor; see, e.g., [1–3]. These arguments are essentially anchored on a simple observation that, since the speed of light is finite, it can take enough time for a signal to propagate from substantially spatially separated points (locations of the servers involved in trading) on Earth potentially to make an arbitrageable difference. We will not delve into this train of thought here. For other works alluding to relativistic effects, see, e.g., [4–6]. We comment on the last reference at the end of Section 5.

4 For a partial list, see [11–27], and also references contained therein.

5 See, e.g., [28–35], and references therein.

6 For physics related works, see, for instance, [36–47], and references therein.

7 For relativistic heat conduction, see, e.g., [48–50].

8 More precisely, the telegraph equation has a "wrong-signed" (for our purposes here, that is) second-order time derivative. In fact, as we discuss below, second-order (in time derivatives) equations are not suitable for pricing purposes. There should be only a single time derivative.

9 Not to be confused with a "Gaussian approximation" mentioned below. This trick is exact.

10 The aforesaid constant interest rate is then given by $r_0 = E_0/\hbar$ ($\hbar$ is the reduced Planck constant) and the map to finance is via identifying $\hbar/m$ with σ^2 (σ is the diffusion coefficient—see below). The financial interpretation of c is given below. It is *not* a maximal speed (cf. physics).