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Equilibrium distributions in entropy driven balanced processes

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HIGHLIGHTS

- Entropy driven equilibrium distributions are found by counting of states.
- We show the exclusiveness of the Polya, Bernoulli, Negative Binomial and Poisson distributions.
- A general master equation framework is given for the evolution of network connectivity and particle production.
- For dynamics satisfying detailed balance we prove the decrease of the generalized entropic distance.

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ABSTRACT

For entropy driven balanced processes we obtain final states with Poisson, Bernoulli, negative binomial and Pólya distributions. We apply this both for complex networks and particle production. For random networks we follow the evolution of the degree distribution, P_n, in a system where a node can activate k fixed connections from K possible partnerships among all nodes. The total number of connections, N, is also fixed. For particle physics problems P_n is the probability of having n particles (or other quanta) distributed among k states (phase space cells) while altogether a fixed number of N particles reside on K states.

1. Introduction

Stationary distributions, as results of entropy driven processes are dominated by phase space factors in contrast to "energy driven" processes, where the final state is more determined by the interaction (potential) energy than the kinetic degrees of freedom. Entropy driven processes are typical at high temperatures, where the entropic contribution to the free energy is increasingly important. Having in mind a statistical–thermodynamic approach to the hadronization problem in high-energy physics, models emphasizing the phase space factor are useful, while approaches based on interaction matrix elements are more purposeful for describing branching ratios in unidirectional decay chain processes. In this paper we deal with a model framework considering entropy driven processes and their limiting distributions typical for high-temperature, near-equilibrium situations. The competing view of far-equilibrium, unbalanced processes is postponed to a later publication [1].

Balanced processes on the other hand include all evolutions where a transition and its reverse between the states of a system are both allowed, albeit sometimes with different rates, if in the microscopic processes a preference is built in.

The general laws of equilibrium and near-equilibrium thermodynamics are classical knowledge. The existence of macro-equilibrium based on micro-dynamics and its stability properties are closely connected with the physical notions of

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temperature, heat and entropy. There are, however, still some open problems left to modern statistical physics. Generalizations of the entropy –probability connection [2–8], beyond mathematical games, also require the re-interpretation of the notion of equilibrium [9] and the composition rules for uniting smaller systems in bigger and more complex ones [10,11]. Also questions, related to far from equilibrium behavior of large dynamical systems, like growing networks [12–18], are intriguing.

Here we consider a unified approach to all statistics resulting from a balanced micro-dynamics applicable to a wide class of physical models. In particular we discuss the case of randomly connected networks and randomly produced particles with some imposed conservation laws.

For processes near equilibrium typically a subsystem and a reservoir exchange physical currents in a locally symmetric and microscopically reversible way, establishing in due of time a detailed balance. This state is characterized then by the distribution of those conserved quantities. Our first example is the hadronization process: In high-energy accelerator experiments the number of created particles per event fluctuates. Since the total energy is fixed in such experiments, the distribution of hadron numbers from one collision event to another determines the effective thermal-like properties of the observed kinetic spectra, [8,19].

Another example is given by random networks, where due to a balance between growth and decays of the connections the degree distribution tends to a few particular shapes in equilibrium. Such studies have become popular in the last decades [20–25]. Random networks are characterized by the probability distribution, P_n, of having a given number of links, n, known as the degree distribution. The connection between the indexed nodes, $i = 0, 1, \ldots, k$, can be described by an adjacency matrix, C_{ij} containing zero for no connection and a number for a link pointing from node i to node j. In unweighted networks the entries of C_{ij} are just zeros and ones, and for undirected networks only the upper triangle of the matrix is used. For more general considerations, however, e.g. on directed networks this matrix is not necessarily symmetric, $C_{ji} \neq C_{ij}$. Weighted connections also may be of relevance for some statistical problems, in such cases C_{ij} can be any real number. Even self-connections, $C_{ii} \neq 0$ have to be allowed for the most general network.

In this view a random network is a random ensemble of C_{ij} values. It looks analogous to a rectangular box with altogether $K = k \times k$ cells, onto which N (multiple, including self-) connections are randomly thrown. By an analysis, in particular by asking for the probability of a given multiplicity connection from a single node, one chooses to sum over k cells in a row (or in a column) and asks for the P_n probability for finding exactly n connections. This is similar thus with the case of particle physics problems, where P_n is the probability of having n particles (or other quanta) distributed among k states (phase space cells) while altogether a fixed number of N particles reside on K states.

In a general picture applicable both for particle and link distributions under some conservation constraints, we assume K cells, $N = pK$ "stones" (i.e. units of connection strength). If only 0 or 1 stone can be in a cell then $p < 1$ (*fermionic* systems), for an arbitrary number of multiple connections $p > 1$ is also possible (*bosonic* systems). The stationary "degree distribution" of the nodes is given by the probability that a single row (with k boxes) contains *exactly* n stones $Q_n = \text{Prob}(n, k; N = pK, K = k^2)$.

We present analytic solutions to the above problems in the frameworks of (i) a pure statistical counting and (ii) in a master equation approach.

2. Statistics of random displacements

For a totally random displacement of N particles in K cells and asking for observing n ones in k cells one obtains the distribution following the idea suggested by Boltzmann: the probability of such an observation is given by the ratio of the numbers of arrangements with and without splitting the system to k and $K - k$ cells, respectively. The number of combinations of n particles in k cells, if each cell can be occupied at most by one particle (the *fermionic* case), is given by:

$$W(k, n) = \frac{k!}{n!\,(k-n)!} = \binom{k}{n}. \tag{1}$$

In particle physics fermions behave this way and for networks this result corresponds to unweighted directed links. Networks constrained by $N \leq K$ and $n \leq k$ are *fermionic*. If on the other hand $N > K$ and correspondingly $n > k$ is allowed, then there must be multiple connections. Such systems we label as *bosonic* ones.

The probability of having exactly n particles (connections) in k cells (from maximal k connections) while in a huge system altogether N particles (connections) are randomly distributed in K cells (among the maximal number of partner nodes) allowing only single occupation (single connections) is given by the following Pólya distribution:

$$Q_n = \frac{W(k, n)\,W(K-k, N-n)}{W(K, N)} = \frac{\binom{k}{n}\binom{K-k}{N-n}}{\binom{K}{N}}. \tag{2}$$

This distribution is normalized, $\sum_{n=0}^{\infty} Q_n = 1$. For accessing different limits of the Pólya distribution, we utilize the generic approximation

$$W(K, N) \xrightarrow[K \gg N]{} \frac{K^N}{N!}, \tag{3}$$