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Correlation and relaxation times for a stochastic process with a fat-tailed steady-state distribution

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HIGHLIGHTS

- Relaxation and correlation times are obtained for stochastic Inverse Gamma process.
- Cumulants and their relaxation times diverge due to multiplicativity of IG process.
- Single time scale defines correlations, as in Ornstein–Uhlenbeck process.
- Distribution of relaxation times of entire distribution is Inverse Gaussian.

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ABSTRACT

We study a stochastic process defined by the interaction strength for the return to the mean and a stochastic term proportional to the magnitude of the variable. Its steady-state distribution is the Inverse Gamma distribution, whose power-law tail exponent is determined by the ratio of the interaction strength to stochasticity. Its time-dependence is characterized by a set of discrete times describing relaxation of respective cumulants to their steady-state values. We show that as the progressively lower cumulants diverge with the increase of stochasticity, so do their relaxation times. We analytically evaluate the correlation function and show that it is determined by the longest of these times, namely the inverse interaction strength, which is also the relaxation time of the mean. We also investigate relaxation of the entire distribution to the steady state and the distribution of relaxation times, which we argue to be Inverse Gaussian.

1. Introduction

For a stochastic process with a steady-state distribution, a natural question arises of what is the relaxation time towards the stationary process. In other words, when the initial values of the variables are chosen from a distribution that differs from that of the steady-state distribution, what is the time for the distribution to settle into its steady-state and for various quantities to achieve their stationary values.

Of particular interest is the situation when the steady-state distribution has power-law ("fat") tails and, thus, the diverging cumulants – especially the divergent lowest cumulants: variance or even mean. In the latter circumstance, one needs to devise the means of ascertaining, including numerically, that the steady-state distribution has been reached, especially in the circumstance when the latter may be unknown analytically.

Another point of interest for such processes is that of the relationship between the correlation and relaxation times. A related issue is that of relevant time scales and steady-state distributions in strongly (power-law) correlated time series.

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In this paper we study a stochastic Ito process

$$dx = -J(x-1)dt + \sqrt{2}\sigma x dB \quad (1)$$

with the Inverse Gamma (IGa) steady-state distribution,

$$P_0(x) = \frac{e^{-\frac{J}{x\sigma^2}}\left(\frac{J}{x\sigma^2}\right)^{2+\frac{J}{\sigma^2}}}{\frac{J}{\sigma^2}\Gamma\left(1+\frac{J}{\sigma^2}\right)}, \qquad x \geq 0. \quad (2)$$

This process is a limiting case of the Generalized Inverse Gamma (GIGa) process,[1] which describes a stochastic birth–death model, which appears in diverse contexts, such as generalized Bouchaud–Mézard (BM) network model of economic exchange [1–3], ontogenetic mass distribution [4,5] and market volatility [6]. The first term in Eq. (1) describes the reversion to the (unit) mean characterized by the interaction strength $J > 0$[2] and dB in the second term is the Wiener term, with $\sigma > 0$ characterizing stochasticity.

The main time dependence of the cumulants of the time-dependent distribution is given by (Section 3)

$$\kappa_n(t) \propto \frac{1-e^{-\lambda_n t}}{\lambda_n} \quad (3)$$

where (Section 2)

$$\lambda_n = n[J - \sigma^2(n-1)], \qquad 1 \leq n \leq 1 + J/\sigma^2 \quad (4)$$

which is predicated, of course, on the assumption that the initial values are not chosen from the steady-state distribution (in particular, $\kappa_n(0) \neq \kappa_n(\infty)$ for all n, the latter being the nth cumulant of the steady-state distribution), since in such case the process is stationary. As stochasticity increases via σ^2, progressively lower cumulants become divergent, $\kappa_n \propto \lambda_n^{-1}$ as $\sigma^2(n-1) \to J$ for progressively smaller n, as do their respective relaxation time $\tau_{relax}^{(n)} \sim \lambda_n^{-1}$. Once $\sigma^2 > J$, cumulants of the steady-state distribution no longer exist with the exception of the mean, whose relaxation time is $\tau_{relax}^{(1)} \sim \lambda_1^{-1} \sim J^{-1}$.[3]

As will be discussed later (end of Section 2), for $J > \sigma^2$, the IGa process is characterized by the correlation function

$$\langle \delta x(t+\tau)\delta x(t)\rangle = \frac{e^{-J\tau}(1-e^{-2(J-\sigma^2)t})}{\frac{J}{\sigma^2}-1} \quad (5)$$

and becomes divergent as $\sigma^2 \geq J$ (conversely, for $J \gg \sigma^2$, when we can set $x \approx 1$ in the stochastic term, we recover the well-known correlation function of the Ornstein–Uhlenbeck process). This follows from the eigenvalue analysis of the Fokker–Planck (FP) equation [8–10]. In this formalism, the eigenvalues are given by Eq. (4), but only half of them correspond to a complete set of orthogonal eigenfunctions that have $P_0(x)$ as their "attractor",

$$\lambda_n = n[J - \sigma^2(n-1)], \qquad 1 \leq n \leq \frac{1+J/\sigma^2}{2}. \quad (6)$$

The goal of this paper is to examine the relaxation towards steady-state distribution, especially on approach to and in the regime when $\sigma^2 > J$. In Section 2, we discuss the analytical eigenvalue formalism and, in particular, the correlation function (5). In Section 3, we derive and numerically examine the cumulant relaxation. In Section 4, we study relaxation of the distribution as a whole to IGa and argue that relaxation times generated by (1) along different paths are distributed as Inverse Gaussian (IG).

2. Eigenvalue formalism for stochastic IGa process

The FP equation for the stochastic IGa process (1) can be written as

$$\frac{\partial P(x,t)}{\partial t} = J\frac{\partial[(x-1)P(x,t)]}{\partial x} + \sigma^2\frac{\partial^2[x^2P(x,t)]}{\partial x^2}. \quad (7)$$

We seek solution in the standard form [8]:

$$P(x,t) = P_0(x) + P(\lambda;x)e^{-\lambda t} \quad (8)$$

[1] Properties of GIGa function, the steady-state distribution for the GIGa process are discussed in some detail in Appendix B of Ref. [6]; see also Refs. (15–18) in Ref. [3].

[2] A multiplicative geometric Brownian process with mean repulsion, $J < 0$, does not have a steady-state distribution but may still be useful for economic models.

[3] A somewhat analogous divergence of moments in a circumstance where the steady-state distribution is power-law is described in [7].