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On quantitatively measuring controllability of complex networks

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HIGHLIGHTS

- Based on an index, the controllability of complex networks can be measured quantitatively.
- Studies on random networks, small-world networks, and scale-free networks are performed respectively.
- A possible way for comparing the relative controllability of different network topologies is discussed.

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ABSTRACT

This paper deals with controllability of complex networks. An index is chosen to quantitatively measure controllability of given network. The effect of this index is analyzed mainly based on empirical studies on various classes of network topologies, such as random network, small-world network, and scale-free network. Such an endeavor could help to enable the comparison of controllability between different network topologies.

1. Introduction

The controllability of a dynamical system reflects the ability of external input information to influence the motion of the overall system. Observability and controllability are dual alternatives. Integrated with stability, they form the theoretical foundation for most of the systems analysis and synthesis problems. Therefore, controllability has been one of the most important concepts in modern control theory.

Since about a decade ago, the controllability problems of dynamical networked large-scale systems have intrigued many scholars from both the control [1–7] and the physics [8–15] communities, and are expected to attract the attention of broader disciplines.

Tanner [1] studied the controllability of systems with a single leader and conjectured that excessive connectivity may be detrimental to controllability, while giving a definition of graph controllability based on a partition of the associated Laplacian matrix. Paying attention to the relationship between the extent of graph symmetry and controllability, Rahmani and Mesbahi [2] further extended the results in [1]. Cai et al. [3,4] addressed the controllability problems of a class of high-order systems, proposing a scheme of controllability improvement. Liu et al. [5] concerned the controllability of discrete-time systems with switching graph topologies. Ji et al. [6,7] dealt with the interactive protocols, endeavoring to integrate the influence of three facets upon controllability, which are the protocol, the vertex dynamics and the network topology.

Liu et al. [8] addressed the structural controllability of complex networks. They selected an index denoted by N_D to quantitatively measure the extent of controllability of complex networks, namely the least amount of independent input

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signals required. Along the route of [8], there have emerged a number of papers from the physics community, e.g. [9–15]. Particularly, Yan et al. [9] concentrated upon the problem of minimal energy cost for maneuvering the nodes. Yuan et al. [10] concerned the exact controllability of undirected networks with identical edge weights and discovered certain consistency in N_D between structural controllability and exact controllability. Sun and Motter [11] discovered a fact that even the systems that pass the Kalman controllability criteria may still be uncontrollable in practice due to numerical effects, which was later substantiated in [12].

The concept of controllability for dynamical systems was originally raised by Kalman, with a set of algebraic criteria to check whether or not a given system is controllable. It has formed the foundation of the controllability theory. However, there are two intrinsic issues somewhat limiting the study of the controllability of complex networks from the viewpoint of Kalman controllability. The first issue is that almost any arbitrary system is completely controllable in the sense of Kalman controllability. This fact reduces the significance of being controllable. As the second issue, it is rather difficult to translate those algebraic criteria into straightforward conditions for the topologies of networks.

In comparison with Kalman controllability, the concept of structural controllability possesses some advantages for coping with the controllability problems of networked systems. First, unlike Kalman controllability, being structurally controllable or not is essentially distinct for any configuration. Second, it is possible to acquire concise and straightforward criteria on the topologies of networks to check whether or not they are structurally controllable. Nevertheless, the essence of structural controllability is only a minimal requirement for the availability of input information across the overall network, which is just a necessary prerequisite for control. Actually, this conception could hardly facilitate evaluating the efficiency of control. Therefore, structural controllability is also restrictive.

In the current paper, a third angle on controllability is addressed. We shall endeavor to explore the possible ways to quantitatively measure the extent of controllability of any given network. It is motivated by a wish to overcome some of the limitations of existing controllability concepts, and to extend the methodology for controllability analysis of networks, from qualitative to quantitative. An index will be proposed for evaluating the potential for a dynamical network to be easily controlled via the input information. Simulations will be performed on three distinct types of complex networks, namely the E–R networks, the WS small-world networks, and the BA scale-free networks, to illustrate the diversity of controllability situations.

The remaining part of this paper is organized as follows. Section 2 introduces the fundamental preliminaries about controllability of complex networks and describes the model concerned. Section 3 endeavors to analyze the computational controllability concept theoretically, based on some simple examples. Section 4 presents a series of experimental results from three typical complex networks. Finally, Section 5 concludes this paper.

2. Problem description and preliminaries

2.1. Kalman controllability

The time-varying dynamics of a network concerned throughout this paper with N_f followers and N_l leaders is described by the following equation:

$$\dot{\xi} = \frac{d\xi}{dt} = A_{ff}\xi + A_{fl}\eta \tag{1}$$

where the vector $\xi(t) \in R^{N_f}$ represents the states of the follower vertices; $\eta(t) \in R^{N_l}$ represents the states of the leader vertices, which can be determined externally; and $A \in R^{(N_f+N_l)\times(N_f+N_l)}$ is the weighted adjacency matrix of the overall network topology, which is decomposed as follows according to the leader–follower distribution

$$A = \begin{bmatrix} A_{ff} & A_{fl} \\ A_{lf} & A_{ll} \end{bmatrix}.$$

The network is controllable in the sense of Kalman if the state values of the followers can be completely controlled via appropriately selected state values of leaders, otherwise, it is uncontrollable. This conforms to the conventional definition of Kalman controllability [16].

Definition 1. A dynamical network (1) is completely controllable if for any initial values of follower states

$$\xi_1(0), \xi_2(0), \ldots, \xi_{N_f}(0) \in R$$

there exist $\tau < \infty$ and proper leader signals

$$\eta_1(t), \eta_2(t), \ldots, \eta_{N_l}(t) \quad (t \in [0, \tau])$$

such that $\xi_1(\tau) = \xi_2(\tau) = \cdots = \xi_{N_l}(\tau) = 0$.

Lemma 1 below provides the most fundamental criterion to check controllability, known as the rank test.