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Periodic sequences of simple maps can support chaos

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HIGHLIGHTS

- We show that the combination of simple maps may produce chaos.
- Chaos may depend on the order they are taken.
- Chaos appears easily if the number of maps increases.

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ABSTRACT

In this paper, we explore the Parrondo's paradox when several dynamically simple maps are combined in a periodic way, producing chaotic dynamics. We show that the paradox is not commutative, that is, it depends on the way that the maps are iterated. We also see that the paradox happens more frequently when the number of maps that we iterate increases.

1. Introduction

The Parrondo's paradox was introduced in the setting of game theory, providing examples of losing (resp. winning) games which are combined to construct a winning (resp. losing) game (see Refs. [1–4]). This paradox can be reformulated in the setting of discrete dynamical systems, which can be called the dynamic Parrondo's paradox, in terms of "chaos + chaos = order" or "order + order = chaos" (see e.g. Refs. [5–8] for some examples). Additionally, periodic sequences of maps have been recently used to construct some population dynamics models (see e.g. Refs. [9–11]).

If the phase space is an interval on the real line $I = [a, b]$, $a < b$, the paradox consists of two continuous maps $f, g : I \to I$. Then, we construct a sequence of maps $f_{1,\infty} = (f_1, f_2, \dots)$ such that f_i is either f or g for $i \geq 1$. There exists paradox if the maps f and g are dynamically simple (resp. complex or chaotic) and the sequence $f_{1,\infty}$ generates chaotic (resp. simple) dynamics. The above sequence $f_{1,\infty}$ can be seen as an orbit of the system $\Phi : \Sigma \times I \to \Sigma \times I$, where $\Sigma = \{(s_n)_{n=0}^{\infty} : s_n \in \{0, 1\}\}$, and for $(s_n)_{n=0}^{\infty} \in \Sigma$ and $x \in I$,

$$\Phi((s_n)_{n=0}^{\infty}, x) = (\sigma((s_n)_{n=0}^{\infty}), f_{s_0}(x)),$$

where $\sigma((s_n)_{n=0}^{\infty}) = (s_{n+1})_{n=0}^{\infty}$ is the shift map, and $f_{s_0}(x) = f(x)$ if $s_0 = 0$ and $f_{s_0}(x) = g(x)$ otherwise. For instance, if the sequence $s_n = (0, 1, 1, 1, 0, 1, \dots.)$, then the sequence $f_{1,\infty} = (f, g, g, g, f, g, \dots)$ and a trajectory at a point $x \in [0, 1]$ is $(x, f(x), g(f(x)), g(g(f(x))), \dots.)$.

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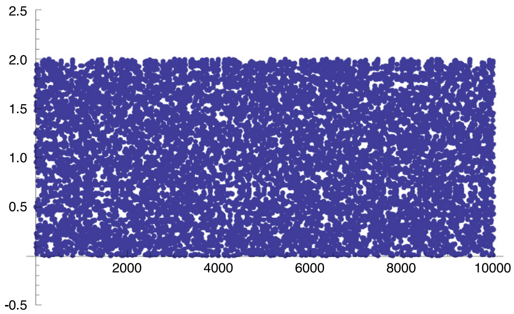

Fig. 1. Time series produced by iterating two linear maps $f(x) = x/2 + 1$ and $g(x) = -x/2 + 1$ in a random way. Note that although the maps are defined on the whole real line, the orbit seems to be restricted in a compact subinterval.

If the sequence $(s_n)_{n=0}^{\infty}$ can be freely chosen, then two affine maps defined on the real line with two different fixed points produce trajectories which apparently have a chaotic dynamical behavior. Each single map produces an orbit which is dynamically simple, but when we combine them randomly, we can produce orbits that seem to be unpredictable. For instance, we chose $(s_n)_{n=0}^{\infty}$ in a random way and take the affine maps $f(x) = x/2 + 1$ and $g(x) = -x/2 + 1$. Then, Fig. 1 shows the time series of a typical orbit.

It is proved in Refs. [12–14] that the topological entropy of Φ, which is a measure of its complexity (see Refs. [15,16]), agrees with the topological entropy of the shift map σ. This implies that the dynamics of the different sequences $f_{1,\infty}$ do not produce additional complexity and then, the complexity is given by the shift map σ.

If the sequence $(s_n)_{n=0}^{\infty}$ is periodic, then two affine maps as above cannot produce a chaotic time series. However, if we consider two continuous interval maps, the existence of paradox of both types, "chaos + chaos = simplicity" and "simplicity + simplicity = chaos", given by the sequence $(0, 1, 0, 1, \ldots)$, is shown in several papers [5,6,8,17].

In this paper, we wonder how the paradox evolves when the period of the sequence $(s_n)_{n=0}^{\infty}$ increases. To analyze this question, we consider the well-known logistic family $f_a(x) = ax(1 - x)$, $a \in [1, 4]$, and study when switching periodically simple maps may produce chaotic dynamics. This question has been partially analyzed in Ref. [18], where the topological entropy of a sequence of three non-chaotic maps has been computed with prescribed accuracy. However, it is not possible to go further of period three (for $(s_n)_{n=0}^{\infty}$) when we compute the topological entropy with the algorithm introduced in Ref. [18]. Then, we consider the well-known Lyapunov exponent to analyze when non-chaotic maps combined in a periodic way can produce chaos. As a first interesting result, we show that the existence of paradox depends on the way the maps f and g are combined to form the periodic sequence $f_{1,\infty}$. In other words, Parrondo's paradox is not commutative.

The paper is organized as follows. In Section 2 we introduce some basic notation and previous results to analyze the main aim of this paper in Section 3.

2. Preliminary results

Let $f_{1,\infty} = (f_n)$ be a sequence of interval maps $f_n : I \to I$. The orbit of a point $x \in X$ under $f_{1,\infty}$, denoted by $\mathrm{Orb}(x, f_{1,\infty})$, is given by the sequence $(f_1^n(x))$, $n \in \mathbb{N}$, where $f_1^n = f_n \circ \cdots \circ f_1$. $f_1^0(x) = x$ for all $x \in I$. Note that, when $f_{1,\infty} = (f_n)$ is a constant sequence, then $f_1^n(x)$ is $f^n(x)$ and the sequence $f_{1,\infty}$ defines a classical discrete dynamical system. If $f_{1,\infty}$ is periodic of period k, that is, k is the smallest positive integer such that $f_{n+k} = f_n$ for all $n \in \mathbb{N}$, then $f_{1,\infty}$ is characterized by the first k elements, and then we denote $f_{1,\infty} = [f_1, \ldots, f_k]$. For instance, we consider in this paper the well-known logistic family $f_a(x) = ax(1 - x)$. If for instance we consider the values $a = 3.5$ and 4, then we construct the periodic sequence $[f_{3.5}, f_4]$, which is given by the sequence $(f_{3.5}, f_4, f_{3.5}, f_4, \ldots)$. Fig. 2 shows some graphs of this sequence.

Note that

$$\mathrm{Orb}(x, [f_1, \ldots, f_k]) = \mathrm{Orb}(x, f_k \circ \cdots \circ f_1) \cup \cdots \cup \mathrm{Orb}((f_{k-1} \circ \cdots \circ f_1)(x), f_{k-1} \circ \cdots \circ f_1 \circ f_k), \tag{1}$$

and hence the dynamics of $f_{1,\infty}$ can be analyzed by that of the maps $f_k \circ \cdots \circ f_1, \ldots, f_{k-1} \circ \cdots \circ f_{a_1} \circ f_k$. In addition, it is easy to see $\mathrm{Orb}(x, f_k \circ \cdots \circ f_1)$ is periodic if and only if $\mathrm{Orb}((f_{i-1} \circ \cdots \circ f_1)(x), f_{i-1} \circ \cdots \circ f_1 \circ f_k \circ \cdots \circ f_i)$ is periodic for $i = 2, \ldots, k$.

Now, we introduce some basic definitions on interval maps (see e.g. Ref. [16]). A continuous map $f : I \to I$ is said to be l modal if there exist $x_1, \ldots, x_l \in I$ such that $f|_{[x_i, x_{i+1}]}$ is strictly monotone for $i = 1, 2, \ldots, l$. Given $x \in I$, define its ω-limit set $\omega(x, f)$ as the set of limits points of its orbit. Recall that a metric attractor is a subset $A \subset I$ such that $f(A) \subseteq A$, $O(A) = \{x : \omega(x, f) \subset A\}$ has positive Lebesgue measure, and there is no proper subset $A' \subsetneq A$ with the same properties. $O(A)$ is called the *basin* of the attractor. By Ref. [19], the regularity properties of f imply that there are three possibilities for its metric attractors:

1. A periodic orbit (recall that x is periodic if $f^n(x) = x$ for some $n \in \mathbb{N}$).
2. A solenoidal attractor, which is basically a Cantor set in which the dynamics is quasi periodic. More precisely, the dynamics on the attractor is conjugated to a minimal translation, in which each orbit is dense on the attractor. The