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Synchronisation of networked Kuramoto oscillators under stable Lévy noise

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HIGHLIGHTS

- We apply stable Lévy noise to the Kuramoto model of equal frequency oscillators.
- Barabási–Albert and Erdös–Rényi random network cases of 1000 nodes are compared.
- Differences in synchronisation for the two cases generalise beyond Gaussian noise.
- New types of synchrony are seen showing drift depending on the Levy alpha index.
- The results are analytically explained with the fractional Fokker–Planck equation.

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ABSTRACT

We study the Kuramoto model on several classes of network topologies examining the dynamics under the influence of Lévy noise. Such noise exhibits heavier tails than Gaussian and allows us to understand how 'shocks' influence the individual oscillator and collective system behaviour. Skewed α-stable Lévy noise, equivalent to fractional diffusion perturbations, are considered. We perform numerical simulations for Erdős–Rényi (ER) and Barabási–Albert (BA) scale free networks of size $N = 1000$ while varying the Lévy index α for the noise. We find that synchrony now assumes a surprising variety of forms, not seen for Gaussian-type noise, and changing with α: a noise-generated drift, a smooth α dependence of the point of cross-over of ER and BA networks in the degree of synchronisation, and a severe loss of synchronisation at low values of α. We also show that this robustness of the BA network across most values of α can also be understood as a consequence of the Laplacian of the graph working within the fractional Fokker–Planck equation of the linearised system, close to synchrony, with both eigenvalues and eigenvectors alternately contributing in different regimes of α.

1. Introduction

The ability of a network of coupled entities to achieve a synchronised state is a fundamental phenomena observed in numerous complex dynamical systems in areas ranging from physical, chemical, and biological to social. The stylised dynamical model proposed by Kuramoto [1] combines a number of simple features yet exhibits rich behaviours that model real-world situations; see Refs. [2–6] for recent reviews. A number of papers have studied the effect of various forms of white or coloured noise on the Kuramoto dynamics [3,7–12]. In this paper, we show how stable Lévy noise, that has tails

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heavier than Gaussian, influences synchronisation of coupled entities where the network interconnectivity is modelled by various classes of graphs. Such a process allows us to understand how ‘shocks’ influence these well-studied dynamics. There is a close connection between fractional diffusion [13,14] and the class of Lévy processes that we consider; see also Ref. [15]. Therefore, one may also say that we study the influence of fractional diffusion perturbations on synchronisability. We find novel behaviours, such as noise-driven drift and strong breakdown of synchronisation, not seen in previous studies of noise on the Kuramoto model. All these behaviours arise from the heavier tails in the stable Lévy probability distribution that, in turn, induces jumps in the paths of the oscillator dynamics.

Typical studies of the Kuramoto model (without noise) examine a variety of network topologies [16–21] and have established that the ‘scale free’ Barabási–Albert (BA) graphs [22] show the onset of synchronisation at lower coupling than Erdös–Rényi (ER) graphs while the latter approach synchronisation more sharply at the critical coupling. These papers thus demonstrate that the topology of the network plays an important role in synchronisation. A follow-on question is the robustness of this synchronisation when the dynamics are exposed to noise. Many early papers examined the influence of Gaussian noise in a mean field version of the $N \to \infty$ model on a complete graph (reviewed extensively in Ref. [3]). The case of a finite system is studied on a complete graph [7,9] and a ring graph [11] under Gaussian white or coloured noise. In Ref. [12], the finite system is examined on a variety of complex random networks with Gaussian white noise—where the object of study was the generic manifestation of scaling and regression fits across network topologies. Two works [8,10] perturb the Kuramoto dynamics by uniform random noise and compare synchronisation behaviour in the BA and ER cases for zero frequencies. A key observation is that there are differences in the dynamics between the two networks.

We seek to understand how this distinction between BA and ER networks changes as we smoothly move away from Gaussian tails in the noise distribution to one with heavier tails. In particular we are able to confirm the general pattern of dependence of synchronisation on noise strength known for Brownian types of noise. Our application of stable Lévy noise to the Kuramoto model is part of a body of work generalising dynamical systems to the fractional diffusion setting. Other examples already studied in the literature include transport in frictional media [23], one-dimensional non-linear oscillators [24], ratchet potentials [25] and Lotka–Volterra predator–prey models [26]—to name a few from the review paper [13]. These studies have predominately been in the one-dimensional setting so our application of stable Lévy noise to the Kuramoto model represents one of the first high-dimensional examples of fractional diffusion studied in the literature. The motivation is the usefulness of the Kuramoto model as a representation of social processes or distributed decision-making [27–30] but where human cognitive process, known to be cyclic [31], is not only noisy [32] but also may involve leaps of intuition in complex problems [33] or through priming or recognition of previously experienced patterns [34,35]. Lévy types of noise provide a valuable means of quantifying these otherwise qualitative models. A further application lies in the field of brain research where the presence of heavy tails in distributions of alpha and beta rhythms [36] suggests that the brain may be a complex coupled system that may be stylistically modelled by the version of the Kuramoto model that we present in this paper.

In the context of synchronisation, we generalise known results for the Gaussian case but, significantly, uncover an unexpected change in the variety of forms of synchronisation for the stable Lévy case, specifically though we apply zero-mean noise to the Kuramoto system of identical frequencies we observe a robust pattern of positive drift in the collective behaviour of oscillators. We also present a novel explanation for much of this behaviour using the fractional Fokker–Planck approach, which also provides an alternative explanation for behaviours in the Gaussian limit.

The paper is structured as follows. First we outline our set-up for generating networks and the Lévy noise, followed by the numerical scheme to simulate the noise. We then explore the individual pathwise dynamics for different α, and then the collective dynamics. We then examine the fractional Fokker–Planck equation in the linear approximation. The paper concludes with a summary of key insights and future work.

2. Our set-up

To be precise, in this paper we study the Kuramoto model on an undirected graph perturbed by a stable Lévy process given by the stochastic differential equation

$$d\theta_i(t) = \omega_i dt - \frac{K}{N}\sum_{j=1}^{N} A_{ij}\sin(\theta_i(t) - \theta_j(t))dt + dL_i(t). \quad (1)$$

Here θ_i is a time-dependent phase angle at vertex $v_i \in V$ of the connected undirected graph $\mathcal{G} = (V, E)$ of size N. The topology of the graph is encoded in the adjacency matrix A where $A_{ij} = 1$ for $j \neq i$ when vertices v_i and v_j are connected, zero otherwise, and $A_{ii} = 0$. The ω_i are independent and identically distributed random variables modelling the natural frequency of the oscillator at vertex v_i; we later set these to zero to compare with Refs. [8,10]. The quantity $K > 0$ is a real-valued coupling constant.

2.1. Network topology

We focus on two random networks: the classical Erdős–Rényi (ER) and the Barabási–Albert (BA) scale-free graphs. The former is generated by considering N vertices and M edges so that any two vertices are connected with uniform probability

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