Physica A 469 (2017) 731–739

Contents lists available at ScienceDirect

Physica A

journal homepage: www.elsevier.com/locate/physa

A mathematical structure of the separated variational principles of steady states for multi-forces and multi-currents

Kanzo Okada [1]

1-1-2 C-1441 Yokodai, Tsukubamirai, Ibaraki 300-2358, Japan

HIGHLIGHTS

- A concise summary is given for the nonlinear Onsager theory of Edelen.
- Variational principles are built on the dissipation potential for Onsager fluxes.
- A mathematical structure of the separated variational principles is elucidated.
- Quantum Hall effect is analyzed in the framework of the present work.

ARTICLE INFO

Article history:
Received 10 September 2016
Received in revised form 2 November 2016
Available online 27 November 2016

Keywords:
Entropy production inequality
Nonlinear Onsager fluxes
Decomposition theorem
Dissipation potential
Non-dissipative fluxes
Separated variational principles

ABSTRACT

Separated variational principles of steady states for multi-forces and multi-currents in transport phenomena were recently proposed by Suzuki (Suzuki, 2013) by extending the principle of minimum integrated entropy production for a single force found by the same author (Suzuki, 2013). On the other hand, in non-equilibrium thermodynamics, Edelen (Edelen, 1974) generalized the linear Onsager theory to those irreversible processes with significant thermodynamic forces by means of Onsager fluxes. Onsager fluxes by definition satisfy a nonlinear system of reciprocity relations, vanish in thermodynamic equilibrium, and satisfy the second law of thermodynamics. Each system of Onsager fluxes is derivable from a dissipation potential sometimes called the flux potential. This paper aims to elucidate a mathematical structure of the separated variational principles based on the above work of Edelen.

1. Introduction

The so-called separated variational principles of steady states for multi-forces and multi-currents in transport phenomena were recently proposed by Suzuki [1] by extending the principle of minimum integrated entropy production for a single force found by the same author [2]. The key concept of these variational principles is to integrate in all the intermediate processes from the equilibrium state through the final non-equilibrium steady state so as to correctly describe nonlinear irreversible transport phenomena even in the steady states.

On the other hand, Edelen in his 1974 paper [3] generalized the linear Onsager theory to those irreversible processes with large departures from equilibrium by means of Onsager fluxes. Onsager fluxes by definition satisfy a nonlinear system of reciprocity relations, vanish in thermodynamic equilibrium, and satisfy the second law of thermodynamics. Each system of Onsager fluxes is derivable from a dissipation potential, more precisely the flux potential.

E-mail addresses: okada@imi.kyushu-u.ac.jp, kbokada@gmail.com.

[1] Formerly affiliated with the Institute of Mathematics for Industry, Kyushu University, before retirement in March, 2016.

http://dx.doi.org/10.1016/j.physa.2016.11.136

The primary objective of this paper is to elucidate a mathematical structure of the separated variational principles based on the above work of Edelen. Following the notation, conventions, definitions, and arguments given by Edelen [3], we will first provide a review of the decomposition theorem in which any mapping $\boldsymbol{J}(\boldsymbol{X}, \boldsymbol{\omega})$ of the product space $E_M \times E_p$ of an M-dimensional vector space E_M and a p-dimensional vector space E_p into E_M that is of class C^1 in $\boldsymbol{X}$ and continuous in $\boldsymbol{\omega}$ can be represented as the sum of the gradient of a scalar-valued function $\psi(\boldsymbol{X}, \boldsymbol{\omega})$ and vector-valued function $\boldsymbol{U}(\boldsymbol{X}, \boldsymbol{\omega})$ that is orthogonal to the radial vector $\boldsymbol{X}$, namely, $\boldsymbol{J}(\boldsymbol{X}, \boldsymbol{\omega}) = \nabla_{\boldsymbol{X}}\psi(\boldsymbol{X}, \boldsymbol{\omega}) + \boldsymbol{U}(\boldsymbol{X}, \boldsymbol{\omega})$ with $\boldsymbol{X} \cdot \boldsymbol{U}(\boldsymbol{X}, \boldsymbol{\omega}) = \boldsymbol{0}$. The decomposition theorem will play the central role in establishing our objective. If we write the internal entropy production in bilinear form $\boldsymbol{X} \cdot J(\boldsymbol{X}, \boldsymbol{\omega})$, M-dimensional vectors $\boldsymbol{X}$ and $\boldsymbol{J}(\boldsymbol{X}, \boldsymbol{\omega})$ are identified with the thermodynamic forces and the conjugate thermodynamic fluxes, respectively. All vector-valued functions $\boldsymbol{J}(\boldsymbol{X}, \boldsymbol{\omega})$ that satisfy the second law of thermodynamics will be obtained by apply the decomposition theorem to the second law. When the components of $\boldsymbol{J}(\boldsymbol{X}, \boldsymbol{\omega})$ are assumed to be of class C^2, which is by the way not a strong requirement from a physical point of view, any solution of the second law is shown to satisfy the symmetry relations $\nabla_{\boldsymbol{X}} \wedge \{\boldsymbol{J}(\boldsymbol{X}, \boldsymbol{\omega}) - \boldsymbol{U}(\boldsymbol{X}, \boldsymbol{\omega})\} = \boldsymbol{0}$ in $E_M \times E_p$ where $\wedge$ stands for the wedge product. These symmetry relations reduce to the nonlinear Onsager reciprocal relations referred to earlier only when the non-dissipative part of the thermodynamic fluxes $\boldsymbol{U}(\boldsymbol{X}, \boldsymbol{\omega})$ vanishes, and this is the case if and only if $\boldsymbol{J}(\boldsymbol{X}, \boldsymbol{\omega})$ is uniquely determined by $\boldsymbol{J}(\boldsymbol{X}, \boldsymbol{\omega}) = \nabla_{\boldsymbol{X}}\psi(\boldsymbol{X}, \boldsymbol{\omega})$ and the dissipation potential $\psi(\boldsymbol{X}, \boldsymbol{\omega})$ is given by the line integral of $\boldsymbol{J}(\boldsymbol{X}, \boldsymbol{\omega}) \cdot \boldsymbol{dX}$ along a straight line from the origin of E_M to the point $\boldsymbol{X}$. In this paper we restrict our considerations to the case where $\boldsymbol{U}(\boldsymbol{X}, \boldsymbol{\omega}) = \boldsymbol{0}$ throughout E_M, namely to the case of Onsager fluxes. Once the dissipation potential $\psi(\boldsymbol{X}, \boldsymbol{\omega})$ is constructed, it will be used as the Lagrangian function to formulate variational principles of minimum entropy production for stationary states, finally elucidating a mathematical structure of the separated variational principles proposed by Suzuki [1,2] through the associated Euler equations $\boldsymbol{J}(\boldsymbol{X}, \boldsymbol{\omega}) = \boldsymbol{0}$.

2. Entropy production, fluxes and forces, Onsager fluxes

The internal entropy production σ can be written as a bilinear form

$$\sigma = \sigma(X_\Lambda; J_\Lambda) = X_\Lambda J_\Lambda \tag{1}$$

by means of a choice of an M-dimensional vector $X_\Lambda\ (\Lambda = 1 \sim M)$, whose components are identified with the thermodynamic forces and a resulting identification of $J_\Lambda\ (\Lambda = 1 \sim M)$ as an M-dimensional vector, whose components are the conjugate thermodynamic fluxes. It is to be noted that the choice of X_Λ and J_Λ is not necessarily unique. It is to be understood that the summation convention for repeated capital Greek indices is in use. The space and time dependencies of the thermodynamic forces and fluxes will be suppressed except when it is appropriate to make explicit references to them in the succeeding sections.

It should be kept in mind that in a standard form of entropy production, some of the components of the thermodynamic forces X_Λ are spatial derivatives of other components, for example, velocity field and its spatial derivatives, and some partial strings of $X_\Lambda J_\Lambda$ correspond to inner products of vector-valued or tensor-valued thermodynamic forces and its conjugate thermodynamic fluxes, for example, an inner product of heat flux vector and temperature gradients, that of current flux vector and electric field, or the trace of the product of the dissipative stress tensor and the deformation-rate tensor. Although the standard symbols used to describe the quantities involved in various forms of entropy production function for different materials are convenient to give their immediate physical identifications, we can represent the entropy production in bilinear form as given above for the expedience of mathematical treatment and for generality.

Now we assume constitutive relations of the form:

$$J_\Lambda = j_\Lambda(X_\Sigma, \omega_a), \tag{2}$$

where $\omega_a\ (a = 1 \sim p)$ denote all arguments of the M functions $J_1, \ldots, J_M$ that are not thermodynamic forces, namely the thermostatic state variables such as thermodynamic temperature, thermodynamic pressure, displacement gradients, etc. Thus,

$$\sigma = \sigma(X_\Sigma; j_\Sigma(X_\Gamma, \omega_a)) = \sigma^*(X_\Sigma, \omega_a) = X_\Lambda j_\Lambda(X_\Sigma, \omega_a). \tag{3}$$

This relation defines the entropy production as a function defined on the $(M + p)$-dimensional product space $E_M \times E_p$ with coordinates (X_Σ, ω_a) for each $J_\Lambda = j_\Lambda(X_\Sigma, \omega_a)$. We mention here that all the equations to appear from now onwards hold valid for each admissible vector ω_a and will not make any explicit reference to it.

Since $X_\Lambda = 0$ defines a state that satisfies the thermal condition for internal equilibrium, namely, it is characterized by $\sigma^*|_{eq} = 0$ and $\partial\sigma^*/\partial X_\Lambda|_{eq} = 0$, this latter equation in combination with the derivative of Eq. (3) with respect to X_Λ implies $J_\Lambda|_{eq} = 0$. This means that the thermal equilibrium state will be a stable state since there are no disturbing fluxes in this state.

In non-equilibrium thermodynamics, Edelen [3] generalized the linear Onsager theory to those irreversible processes with significant thermodynamic forces by means of the so-called Onsager fluxes. Onsager fluxes $J_\Lambda = j_\Lambda$, which form a nonlinear system of constitutive relations and generalize the linear Onsager relations, are defined as a system of thermodynamic fluxes that satisfies the following three conditions: