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Analysis of structures convertible to repeated structures using graph products

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ABSTRACT

In this paper an efficient method is presented for the analysis of those structures which can be converted to regular forms. The stiffness matrix for such regular structures can easily be inverted using their eigenvalues and eigenvectors. Many non-regular structures can be converted to regular forms. Here the presented method solves not only all the regular forms but also non-regular forms convertible to regular ones. The efficiency of the method is more significant when it is used in reanalyzing and rehabilitating structures where the stiffness matrix should be inverted in each step.

1. Introduction

In recent years the computational ability of computers has improved considerably. However, the efficient and time-saving methods are still of great interest among researchers. In structural engineering, the analysis of large and complex structures are difficult and time-consuming unless efficient approaches are adopted. Moreover, for reanalyzing and rehabilitating, the methods that need repetitive processes, it becomes more important to avoid the large structures and their corresponding matrices during finding the displacements and internal forces. In addition to being time-consuming, poor converge can be another problem in the analysis of such large structures.

Nowadays, using prefabrication in the building construction has resulted in some structures with regular patterns. Regular patterns are the ones in which the stiffness matrix (Laplacian matrix) follows the rules of the graph products, group theoretical and so on where the eigenpairs or inverse of the matrix can be quickly obtained. In this paper, all repetitive, circular and symmetric forms with a quick solution are considered as regular. Special methods are available to solve regular structures in the work of Kaveh and Salimbahrami [1], Kaveh and Sayarinejad [2,3], Kaveh and Nemati [4], Hasan and Hasan [5], Williams [6,7], Thomas [8], Karpov et al. [9] and Chan et al. [10], among many others. Other methods based on graph products were presented by Kaveh and Rahami [11–14]. Group theoretical methods were developed by Zingoni [15–18], Kaveh and Nikbakht [19], and Kaveh [20]. Nevertheless, most typical structures do not follow a repetitive form. However, almost all these structures can be considered as two regular and irregular parts. Considering a structure as a combination of regular and irregular parts instead of an irregular model, can be similar to a substructuring approach [21–23] in which the substructures are composed of irregular and regular parts. Furthermore, in some irregular structures by adding or removing some members and elements or changing magnitudes of their stiffness, a regular structure can be obtained. This approach is similar to a rehabilitating method [24] in which the main irregular structure has the role of a structure before restoring and the regular structure has the role of the restored structure. When it comes to design and reanalysis [25], the main irregular structure in each step is equal to a regular structure plus the stiffness changes of the structural members happening in each step of design.

In these studies only substructuring, rehabilitating and reanalyzing were discussed. While in the present analysis, the above mentioned methods are applied to serve the new concept of the structures composed of regular and irregular parts or structures convertible to regular forms considering the effects of the changes. Here, instead of forming the stiffness matrix and solving an irregular structure as a whole, it converted to a regular one

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that has an exact-fast solution. Then the effect of the changes is applied. The efficiency and productivity of the method is examined comparing the results to those of the standard methods for different examples.

2. Eigenpairs for the graph products and the solution of linear equations through the eigenpairs

Kronecker products are well described in literature, e.g. see Bellman [26] and Brewer [27]. The necessary and sufficient condition to diagonalize simultaneously the Hermitian matrices $\boldsymbol{A}_1$ and $\boldsymbol{A}_2$ using one orthogonal matrix is provided in [28,29]:

$$\boldsymbol{A}_1\boldsymbol{A}_2 = \boldsymbol{A}_2\boldsymbol{A}_1 \tag{1}$$

Consider a matrix $\boldsymbol{M}$ as the sum of two Kronecker products:

$$\boldsymbol{M} = \boldsymbol{A}_1 \otimes \boldsymbol{B}_1 + \boldsymbol{A}_2 \otimes \boldsymbol{B}_2 \tag{2}$$

Consider $\boldsymbol{P}$ as a matrix which diagonalizes $\boldsymbol{A}_1$ and $\boldsymbol{A}_2$ simultaneously, then $\boldsymbol{M}$ can be block diagonalized by $\boldsymbol{U} = \boldsymbol{P} \otimes \boldsymbol{I}$ that means $\boldsymbol{U}^T\boldsymbol{M}\boldsymbol{U}$ is a block diagonal matrix.

If the property is set, it can be written from Eq. (2):

$$\lambda_{\boldsymbol{M}} = \bigcup_{i=1}^{n} eig(\boldsymbol{M}_i); \quad \boldsymbol{M}_i = \lambda_i(\boldsymbol{A}_1)\boldsymbol{B}_1 + \lambda_i(\boldsymbol{A}_2)\boldsymbol{B}_2 \tag{3}$$

where $\boldsymbol{M}_i$s are blocks of the block diagonal matrix $\boldsymbol{U}^T\boldsymbol{M}\boldsymbol{U}$. Each $\boldsymbol{M}_i$ is of dimension m (because it is a linear summation of the matrices $\boldsymbol{B}_1$ and $\boldsymbol{B}_2$ of the dimension m). Since the matrix $\boldsymbol{U}^T\boldsymbol{M}\boldsymbol{U}$ is block diagonal, its eigenvalues are the union of the eigenvalues of each block $\boldsymbol{M}_i$. There are n blocks ($i = 1,2,\ldots,$n), therefore, $\lambda_{\boldsymbol{M}}$ is a set of nm eigenvalues of the matrix $\boldsymbol{U}^T\boldsymbol{M}\boldsymbol{U}$. The matrices $\boldsymbol{U}^T\boldsymbol{M}\boldsymbol{U}$ and $\boldsymbol{M}$ are similar, and therefore $\lambda_{\boldsymbol{M}}$ is also the set of eigenvalues of the matrix $\boldsymbol{M}$.

In the above equation, the dimension of matrices $\boldsymbol{A}_1$ and $\boldsymbol{A}_2$ is equal to n, and the dimension of matrices $\boldsymbol{B}_1$ and $\boldsymbol{B}_2$ is equal to m.

If the condition of Eq. (1) is fulfilled, then the marices $\boldsymbol{A}_1$ and $\boldsymbol{A}_2$ are diagonalized simultaneously by a vector like u. First, it is assumed $\boldsymbol{A}_1 = \boldsymbol{I}$, so Eq. (3) is:

$$\lambda_{\boldsymbol{M}} = \bigcup_{i=1}^{n} eig(\boldsymbol{M}_i); \ \boldsymbol{M}_i = \boldsymbol{B}_1 + \lambda_i(\boldsymbol{A}_2)\boldsymbol{B}_2 \tag{4}$$

Regarding μ and $\boldsymbol{v}$ as the eigenvalue and eigenvector of $\boldsymbol{M}_i$ respectively, we have

$$(\boldsymbol{B}_1 + \lambda\boldsymbol{B}_2)\boldsymbol{v} = \mu\boldsymbol{v} \tag{5}$$

In the following it is shown that $\boldsymbol{u} \otimes \boldsymbol{v}$ is the eigenvector of $\boldsymbol{M}$. Using the property of the Kronecker product we have:

$$(\boldsymbol{A}_1 \otimes \boldsymbol{B}_1 + \boldsymbol{A}_2 \otimes \boldsymbol{B}_2)(\boldsymbol{u} \otimes \boldsymbol{v}) = (\boldsymbol{A}_1\boldsymbol{u}) \otimes (\boldsymbol{B}_1\boldsymbol{v}) + (\boldsymbol{A}_2\boldsymbol{u}) \otimes (\boldsymbol{B}_2\boldsymbol{v}) \tag{6}$$

Since $\boldsymbol{A}_1 = \boldsymbol{I}$ is assumed, therefore

$$\boldsymbol{A}_1\boldsymbol{u} = \boldsymbol{u}; \quad \boldsymbol{A}_2\boldsymbol{u} = \lambda\boldsymbol{u} \tag{7}$$

According to Eqs. (5) and (8), Eq. (7) will be

$$(\boldsymbol{A}_1 \otimes \boldsymbol{B}_1 + \boldsymbol{A}_2 \otimes \boldsymbol{B}_2)(\boldsymbol{u} \otimes \boldsymbol{v}) = \boldsymbol{u} \otimes (\boldsymbol{B}_1 + \lambda\boldsymbol{B}_2)\boldsymbol{v} = \mu(\boldsymbol{u} \otimes \boldsymbol{v}) \tag{8}$$

This equation shows that $\boldsymbol{u} \otimes \boldsymbol{v}$ is the eigenvector of $\boldsymbol{M}$.

The proof is applicable even if $\boldsymbol{A}_1 \neq \boldsymbol{I}$. Because Eq. (1) is still holds, using $\boldsymbol{QZ}$ transformation the two matrices $\boldsymbol{Q}$ and $\boldsymbol{Z}$ can be found such that

$$\boldsymbol{Q}\boldsymbol{A}_1\boldsymbol{Z} = \boldsymbol{I}; \quad \boldsymbol{Q}\boldsymbol{A}_2\boldsymbol{Z} = \boldsymbol{D} \tag{9}$$

where $\boldsymbol{D}$ is a diagonal matrix which converts $\boldsymbol{A}_1$ to $\boldsymbol{I}$.

The stiffness matrix patterns in regular structures hold the forms $\boldsymbol{F}_n(\boldsymbol{A}_m,\boldsymbol{B}_m,\boldsymbol{C}_m)$ and $\boldsymbol{G}_n(\boldsymbol{A}_m,\boldsymbol{B}_m,\boldsymbol{C}_m)$ defined as:

$$\begin{aligned}
\boldsymbol{F}_n(\boldsymbol{A}_m,\boldsymbol{B}_m,\boldsymbol{C}_m) &= \begin{bmatrix} \boldsymbol{A}_m & \boldsymbol{B}_m & & & \\ \boldsymbol{B}_m & \boldsymbol{C}_m & \boldsymbol{B}_m & & \\ & \boldsymbol{B}_m & \ddots & \ddots & \\ & & \ddots & \boldsymbol{C}_m & \boldsymbol{B}_m \\ & & & \boldsymbol{B}_m & \boldsymbol{A}_m \end{bmatrix}_n \\
\boldsymbol{G}_n(\boldsymbol{A}_m,\boldsymbol{B}_m,\boldsymbol{C}_m) &= \begin{bmatrix} \boldsymbol{A}_m & \boldsymbol{B}_m & & & \boldsymbol{B}_m \\ \boldsymbol{B}_m & \boldsymbol{C}_m & \boldsymbol{B}_m & & \\ & \boldsymbol{B}_m & \ddots & \ddots & \\ & & \ddots & \boldsymbol{C}_m & \boldsymbol{B}_m \\ \boldsymbol{B}_m & & & \boldsymbol{B}_m & \boldsymbol{A}_m \end{bmatrix}_n
\end{aligned} \tag{10}$$

The form $\boldsymbol{F}_n(\boldsymbol{A}_m,\boldsymbol{B}_m,\boldsymbol{C}_m)$ represents the stiffness matrix of a repeated path graph. In such a graph the sectors (substructures) are in a repetitive pattern. If a sector-by-sector ordering is used for the nodes, the pattern $\boldsymbol{F}_n(\boldsymbol{A}_m,\boldsymbol{B}_m,\boldsymbol{C}_m)$ will be achieved, where $\boldsymbol{A}_m$ and $\boldsymbol{C}_m$ represent the nodes of each sector and $\boldsymbol{B}_m$ shows that each sector is connected to the next sector. The pattern of $\boldsymbol{A}_m$, $\boldsymbol{B}_m$ and $\boldsymbol{C}_m$ themselves depends on the pattern of the nodes in each sector and the type of their connection to each other that vary from one graph to another. The simplest form of such a pattern can be seen in shear structures, where $\boldsymbol{A}_m$, $\boldsymbol{B}_m$ and $\boldsymbol{C}_m$ change to numbers and the tri-diagonal block matrix gets converted to the tri-diagonal matrix. The form $\boldsymbol{G}_n(\boldsymbol{A}_m,\boldsymbol{B}_m,\boldsymbol{C}_m)$ represents the stiffness matrix of a circulant graph, i.e., a repeated path graph in which the first and last sectors are also connected. The two added blocks $\boldsymbol{B}_m$ in the form $\boldsymbol{G}_n(\boldsymbol{A}_m,\boldsymbol{B}_m,\boldsymbol{C}_m)$, indicate the connection between the first and the last blocks.

The form $\boldsymbol{F}$ with the decomposability condition $\boldsymbol{A} - \boldsymbol{B} = \boldsymbol{C}$ and the form $\boldsymbol{G}$ with the condition $\boldsymbol{A} = \boldsymbol{C}$ can be decomposed and solved rapidly. Then for irregular structures, by adding or reducing some members regular forms will be obtained.

Previously, the inverse of the Laplacian (stiffness) matrix of regular graphs (structures) was obtained using its eigenpairs [30]. Considering $\boldsymbol{Ax} = \boldsymbol{b}$ we have:

$$\{\boldsymbol{\varphi}\}_j^T\boldsymbol{A}\{\boldsymbol{\varphi}\}_j\boldsymbol{y}_j = \lambda_j\boldsymbol{y}_j = \{\boldsymbol{\varphi}\}_j^T\boldsymbol{b} \tag{11}$$

$$\boldsymbol{y}_j = \frac{\boldsymbol{b}_j}{\boldsymbol{\lambda}_j} \Rightarrow \{\boldsymbol{x}\}_n = \sum_{i=1}^{n}\{\boldsymbol{\varphi}\}_i\boldsymbol{y}_i = \sum_{i=1}^{n}\{\boldsymbol{\varphi}\}_i\frac{\boldsymbol{b}_i}{\boldsymbol{\lambda}_i} = \sum_{i=1}^{n}\frac{\{\boldsymbol{\varphi}\}_i\{\boldsymbol{\varphi}\}_i^T}{\boldsymbol{\lambda}_i}\boldsymbol{b} \tag{12}$$

where λ_i and $\{\boldsymbol{\varphi}\}_i$ are the eigenpairs of the matrix $\boldsymbol{A}$.

If we want the stiffness matrix of a repetitive structure to have the form $\boldsymbol{G}_n(\boldsymbol{A}_m,\boldsymbol{B}_m,\boldsymbol{C}_m)$, the Cartesian coordinate system must be changed into an appropriate one. This is because, as an example, in a structure created via rotating an element, the structure has the rotational symmetry property. Then the displacements, caused by a symmetric loading, will be identical in the radial directions. Therefore their projections in x and y directions will not be the same. It means that the displacements in a Cartesian coordinate system can show no symmetry. Therefore, as will be shown in one of the examples, a cylindrical coordinate system will be suitable in such a structure.

3. Converting structures to regular forms and applying the effect of the changes

A method has presented been developed for finding the eigenvalues and eigenvectors of the regular structures. Using these eigenvalues and eigenvectors the equations of the corresponding matrix were solved. The equation $\boldsymbol{F} = \boldsymbol{K}\Delta$ in analysis of regular structures was solved rapidly with high accuracy using this method [30]. Here the irregular structures convertible to regular ones are studied. These kinds of structures are converted to regular structures by utilizing some changes in components. Then the