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Leader selection problem for stochastically forced consensus networks based on matrix differentiation

Leitao Gao [a], Guangshe Zhao [a], Guoqi Li [b,*], Zhaoxu Yang [c]

[a] School of Electronic and Information Engineering, Xi'an Jiaotong University, Xi'an, Shaanxi, 710049, PR China

[b] Center for Brain Inspired Computing Research, Department of Precision Instrument, Tsinghua University, Beijing, 100084, PR China

[c] School of Aerospace, Xi'an Jiaotong University, Xi'an, Shaanxi, 710049, PR China

HIGHLIGHTS

- Leader selection problem in order to minimize the mean-square deviation from consensus in stochastically forced networks is formulated.
- Chain rule of matrix differentiation for obtaining the gradient of the cost function which consists matrix variables is introduced.
- Two algorithms RPGM and PPGM are proposed to solve the formulated optimization problem.
- Convergence properties of RPGM and PPGM are established.

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ABSTRACT

The *leader selection problem* refers to determining a predefined number of agents as leaders in order to minimize the mean-square deviation from consensus in stochastically forced networks. The original leader selection problem is formulated as a non-convex optimization problem where matrix variables are involved. By relaxing the constraints, a convex optimization model can be obtained. By introducing a chain rule of matrix differentiation, we can obtain the gradient of the cost function which consists matrix variables. We develop a "revisited projected gradient method" (RPGM) and a "probabilistic projected gradient method" (PPGM) to solve the two formulated convex and non-convex optimization problems, respectively. The convergence property of both methods is established. For convex optimization model, the global optimal solution can be achieved by RPGM, while for the original non-convex optimization model, a suboptimal solution is achieved by PPGM. Simulation results ranging from the synthetic to real-life networks are provided to show the effectiveness of RPGM and PPGM. This works will deepen the understanding of leader selection problems and enable applications in various real-life distributed control problems.

1. Introduction

Reaching consensus in multi-agent networks has gained much attention in recent years, particularly in distributed systems and social networks [1–4]. Agents in these systems are able to agree on a common value of interests via a local

* Corresponding author.
E-mail address: liguoqi@mail.tsinghua.edu.cn (G. Li).

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interaction protocol. For example, in social networks, a group of individuals are used to agree on a certain value [5,6]. Recently, consensus problems have received considerable attention in the context of distributed control, including the control of unmanned vehicle formations and coordination of mobile autonomous agents [7,8]. When agents in these systems are trying to agree on a certain value, they must maintain this agreement in the face of external disturbances and unreliable communications [9–11]. Thus, it is important to study the robustness of consensus in the networks where all nodes are subject to stochastic disturbances (termed as *stochastically forced networks*) [12–15]. To evaluate a consensus protocol in stochastically forced networks, the communication graph and the mean-square deviation from consensus should be utilized [16–21].

In this paper, we consider undirected consensus networks with two groups of nodes subject to stochastic disturbances. This type of networks are generally formulated as leader-follow systems. In a leader–follower system, the leaders act as external inputs to steer the whole group, and the followers update their states based on the information available from their neighbors. For example, in unmanned vehicle formations control [7], a vehicle is a leader if it has GPS devices, and other vehicles are followers. The local interactions among these vehicles can be used to reach an agreement on heading angle, velocity, inter-vehicular spacing, etc. Thus, to select the vehicle with GPS which can steer the rest of vehicles more effectively becomes significant. This is often referred as the *leader selection problem*. In real-life applications, the leaders may be in the face of external disturbances, which may prevent it from interacting with followers following the desired trajectories. Therefore, it is important to consider leader selection problem in stochastically forced consensus networks.

Leader selection problem has been extensively studied in recent years [21–28]. Leaders were traditionally selected based on majority voting or degree-based approaches [22]. [23] showed that leader selection problem could be relaxed to a semidefinite program and solved efficiently by greedy algorithms (GA). [24] proposed an information-centrality-based formulation that a node in the network graph with maximal information centrality became a leader. The notion of manipulability index in [26] was used to measure the influence of leaders' inputs on the network centroid to select leaders. It has been proven that optimal leader selection in linear multi-agent systems with noisy links was a sub-modular optimization problem [25], which could be solved by minimizing the total error covariance. Then, the lower bound could be obtained by minimizing the mean-square deviation. By using greedy algorithm (GA) to identify leaders, a customized algorithm was developed for large networks [21]. A graph-theoretic approach in [27] showed that an upper bound and a lower bound of the convergence rate to the consensus value were derived based on the maximum distance from leaders to followers. [28] proposed a projected gradient method (PGM) to minimize the cost control and used PGM-extension (PGME) to select key nodes in directed networks.

In this work, we are interested in assigning a predefined number of nodes as leaders in order to minimize the mean-square deviation from consensus in stochastically forced networks. Specifically, we formulate the original leader selection problem as a non-convex optimization model where matrix variables are involved. By relaxing the constraints to a hyperplane, a convex optimization model can also be obtained. Therefore, a key procedure is to obtain the gradient of the cost function. However, it is difficult to take the derivative of a function with respect to a matrix variable which may be depending on other intermediate matrix variables. To this end, we introduce a chain rule [28] to address the issue of matrix differentiation. A projected gradient method (PGM) [28,29] is revisited, namely, revisited PGM (RPGM), to solve the convex optimization problem and the global optimal solution (lower bound) can be obtained. In contrast to greedy algorithm (GA) and PGM-extension (PGME) [28], we develop a probabilistic projected gradient method (PPGM) to select the leaders. We have also applied RPGM, PPGM, GA and PGME to synthetic and real-life networks. Simulation results show that (i) PPGM is simpler than GA but with comparable performance; (ii) PPGM steadily approaches RPGM and outperforms PGME.

This paper is organized as follows. In Section 2, we formulate the leader selection problem and build two models, one is the original non-convex optimization model and the other is a convex optimization model by relaxing the constraints on a hyperplane. In Section 3, we develop the revisited projected gradient method (RPGM) and probabilistic projected gradient method (PPGM) to solve the two proposed optimization problems, respectively. In Section 4, the convergence property of both methods is established. Section 5 contains simulations results in both synthetic and real-life networks. We summary this work in Section 6.

2. Problem formulation

The control objective is to strategically deploy leaders in order to reduce the variance amplification in stochastically forced consensus networks. In this section, we consider a connected, undirected network of N identical agents, with a time-invariant communication structure. The network is modeled by an undirected graph $G = (V, E)$, where V is the set of nodes (with $|V| = N$) and E is the set of edges (with $|E| = m$). The Laplacian matrix [30] of G is

$$L = \widehat{D} - A \tag{1}$$

where $\widehat{D}$ is the diagonal matrix of node degrees and A is the adjacency matrix of G. The Laplacian L is a positive semidefinite matrix.