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Asymptotic behavior of stochastic multi-group epidemic models with distributed delays

Qun Liu [a,b], Daqing Jiang [a,c,d,*], Ningzhong Shi [a], Tasawar Hayat [c,e], Ahmed Alsaedi [c]

[a] School of Mathematics and Statistics, Northeast Normal University, Changchun, Jilin 130024, PR China

[b] School of Mathematics and Information Science, Guangxi Colleges and Universities Key Lab of Complex System Optimization and Large Data Processing, Yulin Normal University, Yulin, Guangxi 537000, PR China

[c] Nonlinear Analysis and Applied Mathematics (NAAM) Research Group, Department of Mathematics, Faculty of Science, King Abdulaziz University, Jeddah 121589, Saudi Arabia

[d] College of Science, China University of Petroleum (East China), Qingdao 266580, PR China

[e] Department of Mathematics, Quaid-i-Azam University 45320, Islamabad 44000, Pakistan

HIGHLIGHTS

- Stochastic multi-group epidemic models with distributed delays are studied.
- We show that there is a unique global positive solution as desired in any population dynamics.
- We show that if $R_0 \leq 1$, the solution of the stochastic system oscillates around E_0.
- We prove that if $R_0 > 1$, the solution of the stochastic model fluctuates around E^*.

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ABSTRACT

In this paper, we introduce stochasticity into multi-group epidemic models with distributed delays and general kernel functions. The stochasticity in the model is a standard technique in stochastic population modeling. When the perturbations are small, by using the method of stochastic Lyapunov functions, we carry out a detailed analysis on the asymptotic behavior of the stochastic model regarding of the basic reproduction number R_0. If $R_0 \leq 1$, the solution of the stochastic system oscillates around the disease-free equilibrium E_0, while if $R_0 > 1$, the solution of the stochastic model fluctuates around the endemic equilibrium E^*. Moreover, we also establish sufficient conditions of these results.

1. Introduction

In the literature of mathematical epidemiology, multi-group epidemic models have been proposed to investigate the transmission dynamics of infectious diseases in heterogeneous host populations, such as measles, mumps and HIV/AIDS and much investigation has been done on many kinds of multi-group models (see e.g. Refs. [1–4]). One of the earliest multi-group models is proposed by Lajmanovich and Yorke [5] for the transmission of gonorrhea. For a class of n-group SIS models, they established the global dynamics and verified the global stability of a unique endemic equilibrium using a quadratic global

* Corresponding author at: College of Science, China University of Petroleum (East China), Qingdao 266580, PR China.
E-mail address: daqingjiang2010@hotmail.com (D. Jiang).

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Lyapunov function. The global stability of the unique endemic equilibrium, which is one of main mathematical challenges in analyzing multi-group models, has been verified through the Lyapunov functional technique. A resolution of this problem has been introduced recently. In Ref. [6], Li et al. characterized a class of multi-group epidemic models with distributed delays which is more general than that in Refs. [7,8], a graph-theoretical approach to the method of Lyapunov functionals was used to establish the global stability of the endemic equilibrium. The host population is divided into several homogeneous groups. They studied the following multi-group epidemic models with distributed delays

$$\begin{cases} S_k' = \Lambda_k - \sum_{j=1}^{n} \beta_{kj} S_k \int_{r=0}^{\infty} f_j(r) E_j(t-r)\mathrm{d}r - d_k^S S_k, \\ E_k' = \sum_{j=1}^{n} \beta_{kj} S_k \int_{r=0}^{\infty} f_j(r) E_j(t-r)\mathrm{d}r - (d_k^E + \epsilon_k) E_k, \quad k = 1, 2, \ldots, n, \end{cases} \tag{1.1}$$

where S_k and E_k represent the susceptible and infected but non-infectious populations in the kth group, respectively. The parameters in system (1.1) have the following biological meanings: Λ_k denotes influx of individuals into the kth group, β_{kj} represents the transmission coefficient between compartments S_k and I_j, d_k^S and d_k^E represent death rates of S and E populations in the kth group, respectively, ϵ_k stands for the rate of becoming infectious after a latent period in the kth group. All parameter values are assumed to be nonnegative and $\Lambda_k, d_k^S, d_k^E > 0$ for all k. Here the kernel function $f_k(r) \geq 0$ is continuous and $\int_{r=0}^{\infty} f_k(r)\mathrm{d}r = a_k > 0$. In system (1.2), the basic reproduction number $R_0 = \rho(M_0)$ (the spectral radius of M_0) determines the disease occurs or not, where $M_0 = \left(\frac{\beta_{kj} S_k^0 a_k}{d_k^E + \epsilon_k}\right)_{n\times n}$, $S_k^0 = \frac{\Lambda_k}{d_k^S}$ and $a_k = \int_{r=0}^{\infty} f_k(r)\mathrm{d}r$, $k = 1, 2, \ldots, n$. If $R_0 \leq 1$, system (1.1) always has the disease-free equilibrium $E_0 = (S_1^0, 0, S_2^0, 0, \ldots, S_n^0, 0)$ and it is globally asymptotically stable in the interior of the feasible region D. It means that the disease dies out from the host population. If $B = (\beta_{kj})$ is irreducible and $R_0 > 1$, then E_0 is unstable and there is an endemic equilibrium $E^* = (S_1^*, E_1^*, S_2^*, E_2^*, \ldots, S_n^*, E_n^*)$ of system (1.1) which is globally asymptotically stable in $intD$, where $D = \{(S_1, E_1, S_2, E_2, \ldots, S_n, E_n) \in \mathbb{R}_+^{2n} : 0 \leq S_k \leq \frac{\Lambda_k}{d_k^S}, 0 \leq S_k + E_k \leq \frac{\Lambda_k}{d_k^*}, k = 1, 2, \ldots, n\}$, $d_k^* = \min\{d_k^S, d_k^E + \epsilon_k\}$, $S_k^*, E_k^* > 0$ and satisfy

$$\Lambda_k = \sum_{j=1}^{n} \beta_{kj} S_k^* a_j E_j^* + d_k^S S_k^*, \quad \sum_{j=1}^{n} \beta_{kj} S_k^* a_j E_j^* = (d_k^E + \epsilon_k) E_k^*.$$

It shows that the disease will prevail and persist in a population.

On the other hand, in real life, epidemic systems are always affected by the environmental noise, which is an important component in an ecosystem. Therefore the deterministic approach has some limitations in modeling the transmission of an infectious disease and it cannot predict the future dynamics of the system accurately. Stochastic differential equation models play an important role in many kinds of applied sciences, including infectious dynamics, since they can provide some additional degree of realism compared to their corresponding deterministic models (see e.g. Refs. [9–16]). As a matter of fact, due to the continuous fluctuations in the environment, the parameters involved in the system do not persist at such a steady-state and always fluctuate around some average values. For better understanding regarding the dynamics of epidemic models, many authors have introduced stochastic perturbations into the deterministic epidemic models to reveal the effect of environmental fluctuations (see e.g. Refs. [17–23]). However, the fraction of papers that obtain asymptotic behavior of stochastic multi-group epidemic models with distributed delays is relatively few.

Motivated by the above works, in this paper, we tend to do some work in this field, our method to include stochastic perturbations is similar to that of Jiang et al. [24]. Here we assume that stochastic perturbations are of the white noise type which are directly proportional to S_k and E_k, influenced on the $\dot{S}_k$ and $\dot{E}_k$ in system (1.1). That is to say, we can replace $-d_k^S$, $-d_k^E$ by $-d_k^S + \alpha_k \dot{B}_{1k}(t)$ and $-d_k^E + \beta_k \dot{B}_{2k}(t)$, respectively, where $\dot{B}_{1k}(t)$ and $\dot{B}_{2k}(t)$ are the white noise, i.e., $B_{1k}(t)$ and $B_{2k}(t)$ are mutually independent standard Brownian motions defined on a complete probability space $(\Omega, \mathcal{F}, \mathbb{P})$ with a filtration $\{\mathcal{F}_t\}_{t\geq 0}$ satisfying the usual conditions (i.e., it is increasing and right continuous while $\mathcal{F}_0$ contains all $\mathbb{P}$-null sets), the intensity of white noise $\alpha_k^2 \geq 0$ and $\beta_k^2 \geq 0$. Then the stochastic version corresponding to system (1.1) takes the following form

$$\begin{cases} \mathrm{d}S_k = \left[\Lambda_k - \sum_{j=1}^{n} \beta_{kj} S_k \int_{r=0}^{\infty} f_j(r) E_j(t-r)\mathrm{d}r - d_k^S S_k\right]\mathrm{d}t + \alpha_k S_k \mathrm{d}B_{1k}(t), \\ \mathrm{d}E_k = \left[\sum_{j=1}^{n} \beta_{kj} S_k \int_{r=0}^{\infty} f_j(r) E_j(t-r)\mathrm{d}r - (d_k^E + \epsilon_k) E_k\right]\mathrm{d}t + \beta_k E_k \mathrm{d}B_{2k}(t). \end{cases} \tag{1.2}$$

The organization of this paper is as follows. In Section 2, we verify that there is a unique global positive solution of system (1.2). In Section 3, we show that if $R_0 \leq 1$, then the solution of system (1.2) oscillates around E_0 of system (1.1) under certain condition. In Section 4, we prove that if $R_0 > 1$, then the solution of system (1.2) goes around E^* of system (1.1) under certain condition. Finally, in order to be selfcontained, we present an Appendix containing some theory used in the previous sections.