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On the average uncertainty for systems with nonlinear coupling

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HIGHLIGHTS

- Nonextensive statistical mechanics is modeled from the perspective of the degree of nonlinear coupling between statistical states.
- The average uncertainty given coupled states is the weighted generalized mean of the probability distribution.
- The density of the coupled exponential distributions at their location plus scale is equal to the matching coupled average uncertainty.
- The coupled logarithm is defined to insure that the unit integral is invariant.
- The coupled entropy, which is an alternative normalization of the Tsallis entropy, is compared with other entropy functions.

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ABSTRACT

The increased uncertainty and complexity of nonlinear systems have motivated investigators to consider generalized approaches to defining an entropy function. New insights are achieved by defining the average uncertainty in the probability domain as a transformation of entropy functions. The Shannon entropy when transformed to the probability domain is the weighted geometric mean of the probabilities. For the exponential and Gaussian distributions, we show that the weighted geometric mean of the distribution is equal to the density of the distribution at the location plus the scale (i.e. at the width of the distribution). The average uncertainty is generalized via the weighted generalized mean, in which the moment is a function of the nonlinear source. Both the Rényi and Tsallis entropies transform to this definition of the generalized average uncertainty in the probability domain. For the generalized Pareto and Student's t-distributions, which are the maximum entropy distributions for these generalized entropies, the appropriate weighted generalized mean also equals the density of the distribution at the location plus scale. A coupled entropy function is proposed, which is equal to the normalized Tsallis entropy divided by one plus the coupling.

1. Introduction

Entropy is the standard-bearer for defining the average uncertainty of a probability distribution or density function [1–3]. Boltzmann, Gibbs, and Shannon (BGS) demonstrated that the kernel $-\ln p$ is necessary to form a weighted arithmetic average of the uncertainty of a probability distribution. Khinchin showed that the entropy function $H = -\sum_{i=1}^{N} p_i \ln p_i$ is unique given the axioms that it is

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(a) continuous in p_i,
(b) maximized at the uniform distribution,
(c) not changed by a state with zero probability, and
(d) additive for conditionally independent probabilities.

Rényi, Tsallis and Amari [4–6] sought to broaden the definition of average uncertainty to account for the influence of nonlinear dynamics in complex systems.
Generalized entropy measures have been applied to a variety of nonlinear systems such as decision making under risk [7,8], communication channel equalization [9], compressive sensing [10], the edge of chaos [11], space plasma [12,13], high energy physics [14,15] and quantum entanglement [16]. Hanel and Thurner [17,18] have shown that requiring just the first three of Khinchin's axioms leads to a two-parameter generalization of entropy, one of which is the Tsallis generalization and the focus of this communication.

The objective of the paper is to show the generalized average uncertainty for a nonlinear system can be defined in the probability domain as the weighted generalized mean. This is derived as a transformation of the generalized entropy functions. New evidence for the importance of generalizing the average uncertainty is provided. For the distributions that minimize the weighted generalized mean (maximum generalized entropy) constrained by a location and scale, the density at the location plus the scale is the generalized average uncertainty.

A review of the average uncertainty for important members of the exponential family provides a helpful framework prior to introduction of the effects of nonlinearity. The Boltzmann–Gibbs–Shannon entropy transformed to the probability domain, is the weighted geometric mean of the distribution, that is

$$P_{avg} \equiv \exp\left(+\sum_{i=1}^{N} p_i \ln p_i\right) = \prod_{i=1}^{N} p_i^{p_i}, \tag{1}$$

where the weights, now as powers, are also probabilities. The *average uncertainty* P_{avg} is the average probability of being able to determine the state of the system. The average uncertainty ranges from certainty ($P_{avg} = 1$) when one of the states is certain, to ($P_{avg} = 1/N$) when all states are equiprobable. The intuition is that the total probability of the distribution is the product of the probabilities and the average is determined by weighting each term by the probability. Assuming a continuous distribution, the average density for the exponential distribution is then

$$\begin{aligned} f_{avg}\left(\tfrac{1}{\sigma}e^{\frac{-(x-\mu)}{\sigma}}\right) &= \exp\left(\int_{\mu}^{\infty} \tfrac{1}{\sigma}e^{\frac{-(x-\mu)}{\sigma}} \ln\left(\tfrac{1}{\sigma}e^{\frac{-(x-\mu)}{\sigma}}\right) \mathrm{d}x\right) \\ &= \frac{1}{\sigma e} \end{aligned} \tag{2}$$

and for the Gaussian distribution the average density is

$$\begin{aligned} f_{avg}\left(\tfrac{1}{\sqrt{2\pi}\sigma}e^{\frac{-1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}\right) &= \exp\left(\int_{-\infty}^{\infty} \tfrac{1}{\sqrt{2\pi}\sigma}e^{\frac{-1}{2}\left(\frac{x-\mu}{\sigma}\right)^2} \ln\left(\tfrac{1}{\sqrt{2\pi}\sigma}e^{\frac{-1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}\right) \mathrm{d}x\right) \\ &= \frac{1}{\sqrt{2\pi e}\sigma}. \end{aligned} \tag{3}$$

In both cases the average density is equal to $f\,(\mu + \sigma)$.

Rényi showed that information theory could be broadened by considering the weighted generalized mean of probabilities [19] as the kernel prior to applying the logarithm function. Through consideration of the statistics of a weighted distribution p_i^q, Tsallis showed that use of both the weighted generalized mean and a generalization of the logarithm function provided a model of non-additive entropy. Because of its role in raising probabilities to a power, the parameter q can be interpreted as the number of independent random variables whose combined distribution provides a basis for determining generalized statistical properties. This analysis, broadly referred to as nonextensive statistical mechanics [20], has been shown to model a wide variety of complex systems.

Section 2 introduces the concept of nonlinear statistical coupling which is an interpretation of nonextensive statistical mechanics focused on the role of nonlinearity in deforming statistical analysis. From examination of multivariate distributions, it was shown that the coupling parameter κ is related to q via the influence of the dimensions d and power α of the state variable by $q = 1 + \frac{\alpha\kappa}{1+d\kappa}$. Other approaches have been taken to parameterizing the Tsallis statistics, including $\kappa' = 1 - q$, which was originally proposed as a definition of nonlinear statistical coupling [21] and has been utilized by other investigators [22,23]; and $\kappa'' = \frac{1}{1-q}$ which is the translation to the kappa-distribution used in space physics [12]. There are also generalizations of entropy, which use the kappa symbol, such as the work by Kaniadakis [24].

In Section 3, the main objective relating average uncertainty and the width of coupled exponential distributions is established. In Section 4 we show that the generalized entropies of Rényi and Tsallis along with a modified normalization of the Tsallis entropy denoted as the coupled entropy, can each be expressed as the weighted generalized mean of the distribution and a transformation to the entropy scale using a generalized logarithm function.