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Static and dynamic consistent perturbation analysis for nonlinear inextensible planar frames

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ABSTRACT

An asymptotically exact method for static and dynamic analysis of geometrically nonlinear planar frames is illustrated. The method is based on an integration of the nonlinear equations for the beam, carried out via a perturbation method, aiming to express the forces at the ends as series expansion of the displacements at the ends and of the distributed loads. Since the beams are assumed to be inextensible and unshearable, also reactive stresses appear among the unknowns, while compatibility conditions must be appended to the equilibrium equations. The element state-relations are assembled for the frame, and discrete, nonlinear perturbation equations are derived. Examples are worked out and relevant results compared with purely numerical solutions.

1. Introduction

Perturbation methods consist in series solutions, able to extrapolate information from linear systems to nonlinear systems, provided nonlinearities are sufficiently weak (typically, when the amplitude of the response is comparatively small). Perturbation methods, of course, cannot compete in accuracy with purely numerical methods, but they offer the strong advantage to supply analytical solutions, in one or more parameters, with a little computational effort, which are particularly useful when the response of a family of systems is sought for, instead of that of a specific system.

Perturbation methods have been widely used to study free and forced nonlinear oscillatory phenomena [1], as well as bifurcation problems, both in static [2] and dynamical fields [3]. While straightforward expansions or the strained coordinate method [4] are sufficient to analyze static or buckling problems, respectively, a richer variety of approaches is likely to be followed in dynamics. Among them, the Multiple Scale Method [4] seems to be the most performing (see e.g. [5] for a discussion); moreover, it appears as the natural counterpart of the 'static perturbation method' [6], as discussed in [7]. The method calls for solving a chain of ode/pde linear equations, each requiring proper solvability conditions, whose combination furnishes "amplitude modulation equations" which govern the slow flow. The method works in lowering the dimension of the original system, thus reducing the original multidimensional dynamic system into a smaller equivalent problem, similarly to what performed by the Center Manifold reduction [8] and Invariant Manifolds theories [9].

Although the algorithm is tailored to an analytical approach (so that innumerable examples have been worked out in literature), numerical-analytical (or semi-analytical) versions have also been proposed. A first attempt was made in [10], where it was stated that "Purely analytical techniques are capable of determining the response of structural elements having simple geometries [...], but they are not applicable to elements with complicated structure and boundaries. Numerical techniques are effective in determining the linear response of complicated structures, but they are not optimal for determining the nonlinear response of even simple elements [...]. Therefore, the optimum is a combined numerical and perturbation technique".

Indeed, numerical-perturbation approaches have extensively been applied in literature (see, e.g. [11–14]). They usually consist in two steps, which combine (a) the FEM, which is used to *formulate* a discrete model, starting from a continuous one; and, (b) the perturbation method, which is applied to *solve* the finite-dimensional problem. The solution, therefore, is affected by two types of errors, one of *local* type, related to discretization adopted in each subdomain constituting the structure, the other of *global* type, related to solution by series of the final problem. Indeed, step (a) could be replaced itself by a *local perturbation solution*, able to express (although only in an asymptotic way) the solution of the field equation. Such an approach would avoid using 'a priori' selected interpolating functions, thus overcoming locking problems, and by making the two kinds of approximations mutually consistent. For this reason, such an approach will be referred here as a 'consistent perturbation analysis'. Moreover, it is important to stress, that

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the adjective ‘numerical’, used here to partially connote the algorithm, should *not* be understood in the usual meaning of ‘approximation’, but rather as an ‘automatic procedure’ to numerically evaluate the coefficients of some series expansions, that would be impossible to perform manually. For these reasons, the procedure should be considered as ‘asymptotically exact’.

This approach, however, does not seem to be thoroughly investigated, yet. An attempt along this line was performed in [15], where, by dealing with continuous beams on sliding supports, an algorithm transforming a nonlinear continuous problem into a nonlinear discrete map was illustrated. Of course the consistent perturbation method can only be applied to systems made of elements whose field equations can be asymptotically integrated for general non-homogeneous boundary conditions, typically one-dimensional elements. However, it is expected that it can be applied also for bi- or three-dimensional elements, in the framework of a semi-variational (Kantorovich) approach, as those used, e.g. in [16,17].

In this paper, the algorithm is detailed for a geometrically nonlinear planar frame, made of inextensible and shear-indeformable beams, whose masses are lumped at their ends. The inextensible model appears physically reasonable and analytically suitable, since it reduces the number of field equations; in contrast, however, it calls for a proper treatment of the axial reactive stress, in the framework of a mixed displacement-stress approach [18,19]. Inextensible frames and pantographic-type structures have also been used in the literature to develop higher-gradient theories of materials endowed with microscopic structures [20,21]. The governing equations are derived via a direct (equilibrium and compatibility) approach, differently from [18,19], where a constrained variational problem was tackled; moreover, more general loading conditions are allowed here.

The paper is organized as follows. In Section 2 the nonlinear field equations for a massless beam are asymptotically solved for prescribed displacements at the ends and longitudinal reactive stress, all taken as independent state-variables for the element. This solution is asymptotically exact up-to third-order in the perturbation parameter, which is a measure of the amplitude of the response. In Section 3 a state-relation for the element is drawn, and the relations are successively assembled for the frame. Here lumped masses are introduced and discrete nonlinear equations are derived, expressing equilibrium at joints and kinematic compatibility for the elements. In Section 4 the global equations are solved asymptotically, namely: (a) a straightforward expansion is used for static problems, in order to express the structural response as a function of a load-parameter; (b) the Multiple Scale Method is applied, to evaluate both the free and forced response of the frame, the latter relevant to harmonic loads in primary resonant conditions. Internal resonances have been so far excluded, but they could be easily accounted for. As a consequence, the response is mono-modal, but the contribution of the passive modes (in the Center Manifold perspective) is accounted for. In Section 5 some examples have been worked out and relevant results illustrated. Results provided by the proposed method are validated by a comparison with a FEM solution of both static and dynamic problems. Finally, in Section 6, some conclusions are drawn. Two Appendixes report computational details.

2. Continuous formulation

A straight beam is considered, as an element of a planar frame. The beam is assumed internally constrained and massless. The continuous problem for the single element is formulated, and asymptotically solved.

2.1. Constrained elastic problem for a single beam

The (static) nonlinear elastic problem for a rectilinear beam, is formulated here. The beam is considered to be axially-inextensible and shear-indeformable, and modeled as an elastic, polar, one-dimensional, internally constrained continuum. The field equilibrium equations, in vector form, turn out to be (Fig. 1):

$$\begin{cases} \mathbf{t}'(s) + \mathbf{b}(s) = \mathbf{0}; \\ \mathbf{m}'(s) + \mathbf{a}_t(s) \times \mathbf{t}(s) = \mathbf{0} \end{cases} \tag{1}$$

where $\mathbf{t}(s)$ and $\mathbf{m}(s)$ are the internal force and couple, of reactive and active nature, respectively; $\mathbf{b}(s)$ is the linear density of the external body forces; $\mathbf{a}_t(s)$ is the unit vector tangent at the actual configuration; s is the curvilinear abscissa (both in the reference and actual configuration); finally, a prime denotes differentiation with respect to s.

By introducing the components with respect to the $(\mathbf{a}_x, \mathbf{a}_y, \mathbf{a}_z)$ element – basis, with $(\mathbf{a}_x, \mathbf{a}_y)$ spanning the plane in which the beam bends, and $\mathbf{a}_x$ aligned with the beam-axis in the reference configuration, it follows that (Fig. 1): $\mathbf{m}(s) = M(s)\mathbf{a}_z$, $\mathbf{t}(s) = R(s)\mathbf{a}_x + S(s)\mathbf{a}_y$, $\mathbf{a}_t(s) = \cos\varphi(s)\mathbf{a}_x + \sin\varphi(s)\mathbf{a}_y$, $\mathbf{b}(s) = b_x(s)\mathbf{a}_x + b_y(s)\mathbf{a}_y$.

So that the equilibrium conditions (1) lead to three scalar equations:

$$\begin{cases} M'(s) + S(s)\cos\varphi(s) - R(s)\sin\varphi(s) = 0, \\ R'(s) + b_x(s) = 0, \\ S'(s) + b_y(s) = 0; \end{cases} \tag{2}$$

The beam undergoes a planar displacement field, where $\mathbf{u}(s) = u(s)\mathbf{a}_x + v(s)\mathbf{a}_y$ is the beam axis translation field and $\varphi(s)$ the rotation field of the sections. The strain-displacement relation expresses the link between the unique strain component, the curvature $\kappa(s)$, and the rotation:

$$\kappa(s) = \varphi'(s) \tag{3}$$

while kinematic compatibility calls for internal geometrical constraints (Fig. 1):

$$u'(s) = \cos\varphi(s) - 1, \quad v'(s) = \sin\varphi(s) \tag{4}$$

The material is assumed to behave as linearly elastic, so that the bending moment and the curvature are related by:

$$M(s) = EI\kappa(s) \tag{5}$$

with EI the (uniform) flexural stiffness.

The elastic problem of the internally constrained beam therefore consists of seven scalar Eqs. (2)–(5), in which $M(s)$, $R(s)$, $S(s)$, $u(s)$, $v(s)$, $\varphi(s)$, $\kappa(s)$ are the scalar unknowns. After combination and partial integration, the problem is recast in the following form:

$$\begin{aligned} & EI\varphi(s)'' + S(s)\cos\varphi(s) - R(s)\sin\varphi(s) = 0 \\ & R(s) = R_B + \int_s^l b_x(s)ds, \quad S(s) = S_B + \int_s^l b_y(s)ds \\ & u(s) = u_A + \int_0^s (\cos\varphi(s) - 1)ds, \quad v(s) = v_A + \int_0^s \sin\varphi(s)ds \end{aligned} \tag{6}$$

where $u_A := u(0)$, $v_A := v(0)$, $R_B := R(l)$, $S_B := S(l)$ are integration constants and l is the beam length. Eq. (6) would supply the solution of the problem once u_A, v_A, R_B, S_B were assigned, together with prescribed rotations φ_A, φ_B at the ends:

$$\varphi(0) = \varphi_A, \quad \varphi(l) = \varphi_B \tag{7}$$

However, when the beam is considered as a frame element, the translations u_B, v_B (equal to those of the attached joint) should be considered as assigned, instead of the reactive internal forces R_B, S_B. In this perspective, it is convenient to consider the reactive