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# Dynamic stiffness elements and their applications for plates using first order shear deformation theory

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ABSTRACT

Dynamic stiffness elements for plates are developed using first order shear deformation theory to carry out exact free vibration analysis of plate assemblies. The analysis has been facilitated by the application of Hamiltonian mechanics and symbolic computation. The Wittrick–Williams algorithm has been used as the solution technique. Results have been extensively validated using published literature for both uniform and non-uniform plates. Some finite element results are also provided. The accuracy and computational efficiency of the method are demonstrated. In the final part of the investigation, significant plate parameters are varied and their subsequent effects on the free vibration characteristics are studied.



## 1. Introduction

Aircraft structures are generally modelled as assemblies of thin-walled structural elements. In particular, the top and bottom skins, torsion box, ribs and webs of the wing are idealised as plates. Thus the free vibration analysis of such structures plays an important role in aircraft design. The analysis facilitates aeroelastic and response analyses. The purpose of this paper is to develop the dynamic stiffness method for an accurate and efficient free vibration analysis of plates and plate assemblies.

The usually adopted finite element method [1] (FEM) is a universal tool in structural analysis which can handle complex structures. With the advent of high speed computing, the tendency to use FEM has increased enormously and many commercially available pre and post processing programs have broadened its appeal, making it simple and straightforward to use. FEM is an approximate method, but it generally converges to the exact solution with increasing number of elements. However, the accuracy of results cannot be always guaranteed. This is particularly true in dynamic analysis at high frequencies when the FEM may become unreliable. Thus, there is, and there will always be a need to use analytical methods based on classical theories, wherever possible, to validate the FEM, provide further insights and importantly, restore confidence in design. One such method is that of the dynamic stiffness method [2–8] (DSM) which gives exact results that are independent of the number of elements used in the analysis. For instance, one single structural element can be used in the DSM to compute any number of natural frequencies to any desired accuracy, which of course, is impossible in the FEM. In DSM [2–8], once initial assumptions about the displacement field have been made, no inaccuracy occurs in the analysis. However, for the fundamental mode there is generally very little discrepancy in the frequencies computed using FEM and DSM, but with increasing mode number, significant differences can arise in both response and stability analyses.

The DSM at present has been developed mainly for one-dimensional elements such as bars and beams [3–9]. This is generally accomplished by using the exact closed form solution of their governing differential equations of motion for harmonic oscillation, and relating a state vector of loads to the corresponding state vector of responses at the nodes. The relationship between the two vectors establishes the frequency dependent dynamic stiffness matrix of the element. There are well established computer programs such as BUNVIS-RG [10] and PFVIBAT [11] which demonstrate the accuracy and computational efficiency of the method. A strong point about DSM is that it has all the essential features of FEM such as coordinate transformation, offset connections, assembly procedure, etc., and yet it retains the exactness of results through the use of exact solution of the governing differential equation. However, the solution techniques for FEM and DSM are different. Unlike the conventional FEM which leads to a linear eigenvalue problem, the DSM leads to a non-linear

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eigenvalue problem which is generally solved by applying the Wittrick–Williams algorithm [12,13].

The development of a dynamic stiffness (DS) matrix for a plate element presents considerable difficulties. Wittrick and Williams [14–17] are probably the earliest investigators who developed DSM for simply supported (SS) plates using classical plate theory (CPT). Their theory was later implemented in a computer program called VICONOPT [9,18] which is well-suited to investigate the free vibration as well as buckling behaviour of aircraft wings idealised as prismatic plate assemblies. An important feature of their research is that explicit expressions for the DS elements for SS anisotropic plates were presented. However, the authors did not include the effects of shear deformation and rotatory inertia in their work, which are important when analysing thick plates. The inadequacy of CPT when investigating the free vibration characteristics of thick plates is well known and any method based on CPT will no-doubt incur errors in modal analysis, particularly at high frequencies. Anderson and Kennedy [19,20] advanced VICONOPT by including the effect of shear deformation in their DS development. The explicit terms of the DS matrix were not obtained by them and the problem was solved numerically.

Many higher order shear deformation theories [21,22] have also been developed for thick plates and composite laminates. Reddy and Phan's higher order plate theory [21] has been used to develop the DS matrix of a plate by Leung and Zhou [23]. Also in this case explicit terms of the DS matrix have not been computed.

Using the first order shear deformation theory [24] (FSDT, generally known as Mindlin plate theory in the literature), this paper advances the aforementioned works [9,14–20,23] by developing DS matrix of thick plates by including the important effects of shear deformation and rotatory inertia. Despite the complexity of the problem as a result of the inclusion of these effects, it has been possible to generate explicit expressions for the DS elements for the first time by using symbolic computation (Mathematica [25]). Explicit terms of the DS matrix are essential for developing a quick and efficient computer program which can study plate assemblies as well as for optimisation purposes. Explicit terms of DS matrix of a plate based on the FSDT have never been published before.

Although the dynamic stiffness (DS) development of a Mindlin plate has apparently not been fully investigated earlier, some related works using classical resolution of differential equations with subsequent imposition of boundary conditions (BC) have been published. Reddy [21,26,27] amongst others, analysed the free vibration behaviour of thick plates with the effects of shear deformation and rotatory inertia. These publications are not focused on frequency dependent DS development as in the present case, but an individual plate on its own was studied. For an individual plate, it is possible to determine natural frequencies and mode shapes by applying boundary conditions and eliminating the constants from the general solutions, without resorting to the development of the dynamic stiffness matrix. This procedure is termed as classical method (CM) in the subsequent text. Clearly such as a procedure cannot be easily extended to deal with plate assemblies and somehow lacks generality. Nevertheless, the results obtained by using the CM are useful comparators for validation purposes. The need to apply DSM principally arises to study free vibration behaviour of complex structures. There are two important advantages of DSM out of many. The first one is that two sides other than the SS ones can be constrained without having the need to reevaluate and eliminate the constants as would be required in the CM. More importantly, the second advantage is that the DSM has the capability to assemble element stiffness matrices of complex structures consisting of plate assemblies. For instance, plates with stringers can be analysed and yet exact results can be obtained.

The current investigation is carried out in following steps. First, the fundamental equations of the CPT and FSDT are briefly summarised and some salient features are discussed (Section 2.1). Secondly, the dynamic stiffness matrix based on the CPT is formulated with and without the effect of rotatory inertia (Sections 2.2.1 and 2.2.2) as a precursor to the development of more advanced FSDT DSM which is dealt with in Section 2.2.3 by using symbolic computation [25]. Subsequent to this development, the assembly procedure and imposition of boundary conditions by suppressing appropriate degrees of freedom (penalty method) are explained in Section 2.3.1. This is followed by Section 2.3.2 which highlights the application of the Wittrick–Williams (WW) algorithm for computation of natural frequencies of thick plates with various boundary conditions. The mode finding technique using the DSM is then reported in Section 2.3.3.

Once the DS matrices using CPT and FSDT have been derived, the results computed from the eigensolution procedure are validated in detail for rectangular plates with two opposite sides simply supported and the others having any generic boundary conditions (BC) which can be in any combination of clamped (C), free (F), or simply supported (SS) (Section 3.1). Section 3.2 essentially gives results for a variable thickness plate for which some comparative results are available in the literature. An insight into the advantages and disadvantages using different methods is given and the future potential of DSM resulting form the current research is highlighted. Finally the paper closes with some concluding remarks.

## 2. Theory

### 2.1. Some basic preliminaries

Fig. 1 shows the notation for displacements and forces for a thick plate in a rectangular Cartesian coordinate system. The displacement field is described for both CPT and FSDT and the corresponding equations of motions with their natural boundary conditions are briefly summarised below. For brevity, only

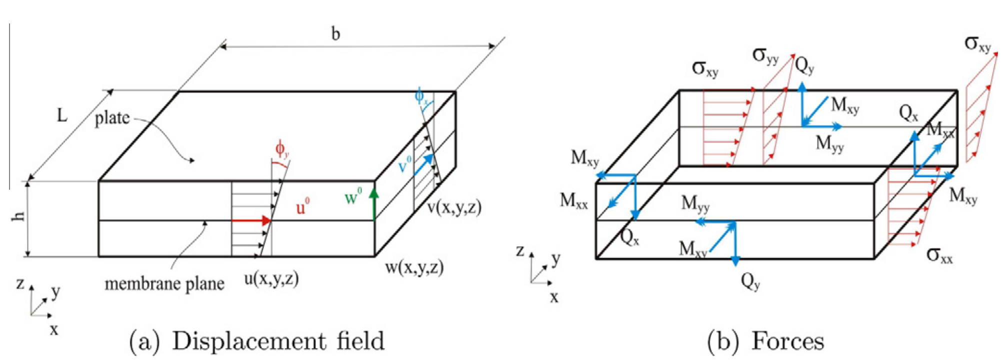


**Fig. 1.** Coordinate system and notations for displacements and forces for a plate.