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Extension of the Sussman–Bathe spline-based hyperelastic model to incompressible transversely isotropic materials

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ABSTRACT

In this paper we extend the Sussman–Bathe spline-based hyperelastic isotropic model to predict the behavior of transversely isotropic isochoric materials. The model is based on an uncoupled decomposition of the stored energy function and a generalization of the inversion formula used by Sussman and Bathe. The present extension introduces some approximations that, in all studied cases, do not yield relevant deviations from the experimental data. The isotropic model results in a particular case of the present formulation. Several possibilities of user-prescribed experimental data are addressed. The model is used to predict experiments of calendered rubber and of biological tissues.

1. Introduction

As it is well known, in the stress and strain analysis of solids there are important practical consequences when considering large deformations. Whereas at small strains the constitutive equations for elastic behavior are simply obtained from the determination of some constants that linearly relate stress and strain components, at large strains the situation is considerably more complex [1]. Materials that achieve large (or moderate) elastic strains behave in a nonlinear manner [2]. Polymers [3] and biological tissues [4] are just some examples. For metals elastic behavior is frequently considered linear in elastoplastic models when using logarithmic stress and strain measures [5,6], but nonlinear when using other strain measures. In order to account for such nonlinearity, specially in computational elastoplasticity, initial formulations for finite element implementation used objective rate (incremental) forms in which the material parameters where constant or a function of the strain state. Stress integration was complex in order to preserve objectivity. In elasto-plasticity, the Rolph–Bathe [7] and Hughes–Winget [8] algorithms are just two examples. As shown by Simó and Pister [9], these formulations are not truly elastic (hence hypoelastic), and energy is not preserved during closed cycles [2]. In order to be elastic, the elastic tensor at large strains has to fulfill some compatibility conditions, apart from full symmetry [10]. Those conditions are automatically accomplished if the constitutive equation is directly obtained from a known stored energy expression or model.

Many such models exist in the literature aimed at the prediction of the behavior of different materials. The Ogden [11], Mooney–Rivlin [12,13], Arruda–Boyce [14], Blatz–Ko [15] and Yeoh [16] models are some of the best known. These models have a given "shape" and some parameters to be determined from a best fit of the experimental data for a given range of expected strains. In a practical problem, the engineer must select the model and the strain range. If fortunate, the predicted behavior may capture the global behavior, but it may miss some finer (probably important) details. Thus one may wonder if having an error on the preserved energy is more important than having an error on the stress–strain behavior if the problem is not of repeated cyclic loading type, sacrificing accuracy for physical and mathematical correctness. Furthermore, what an analyst would like is to just prescribe some stress–strain points and automatically obtain a predicted behavior consistent with the prescribed data. In that sense, there is a temptation to go back to hypoelastic formulations.

A solution to this dilemma has been recently given by Sussman and Bathe [17]. In their work they propose the use of splines to model the stress–strain behavior. The splines are computed such that they pass through experimental stress–strain points and have the desired continuity. Hence, they "exactly" capture the experimental data. Stresses are then expressed in terms of those splines. The key point of their paper is that whereas these initial interpolating splines are not derived from a stored energy function (and hence the behavior is not hyperelastic), the relationship may be inverted. A new set of functions is obtained, which still pass through

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the experimental data and that is derived from a stored energy function (hence the behavior is hyperelastic). Furthermore, computational efficiency is not a relevant issue because the domain of the new set may be divided into equally spaced subdomains so the location within the desired subdomain is a simple operation. The model is a genuine case of WYPIWYG (What You *Prescribe* is What You Get) philosophy which is still physically and mathematically correct.

The Sussman–Bathe model is valid for isotropic hyperelasticity. The purpose of the present work is to extend the model for transversely isotropic materials, as for example some fiber composites, rubber-like materials or biological tissues. We consider the incompressible case, which of course may be extended to the quasi-incompressible case through a volumetric stored energy. Unlike the isotropic model, several cases need to be considered, each case for a given set of possible experimental data. We address those cases in the following sections and also consider isotropy as a special case. Some predictions for actual experiments taken from the literature are presented.

The outline of the paper is as follows. In the following section we review some concepts and procedures essential in our formulation ("building blocks"). Then we can easily introduce the actual procedure on these footings, taking into consideration several possible sets of experiments to obtain the needed data. In the fourth section we show some examples.

2. Building blocks

In this section we briefly review some concepts and formulations that we will use in the procedure outlined in the following section. Once these building blocks are explained and understood, the procedure is relatively simple.

2.1. Splines based piecewise interpolation used in the model

Assume that during a tensile test (or any other test), we have obtained some experimental data given by $N+1$ points of a stress–strain behavior $(y_i \equiv \tilde{\sigma}_i,\ x_i \equiv \widetilde{E}_i),\ i = 1, \ldots, N+1$ in any defined stress and strain measure. We are interested in the interpolation of such data with a given degree of smoothness, see Fig. 1. A handy well known method (specially in CAD) is the use of splines. In our case we will use piecewise cubic splines. In essence the method consists on fitting a third order polynomial between two points such that the slope $y'(x = x_i) \equiv Y_i$ and the derivative of the slope $y''(x = x_i) \equiv Y'_i$ are also the same at both sides of each experimental point. Physically, as seen below, this means that we wish the moduli and its derivative to be continuous, which are attractive smoothness requirements for hyperelastic behavior. In order to obtain the usual tridiagonal system of equations, each subdomain is normalized to $\xi = (x - x_i)/(x_{i+1} - x_i) \in [0, 1]$ so all N polynomia have the expression

$$P_i(\xi \in [0,1]) = a_i + b_i\xi + c_i\xi^2 + d_i\xi^3 \tag{1}$$

For each subdomain, the conditions $y_i = P_i(0)$, $y_{i+1} = P_i(1)$, are given, where y_i are the known experimental data. Between any two subdomains, two additional conditions are established

$$P'_{i-1}(1) = P'_i(0) := Y_i \tag{2}$$
$$P''_{i-1}(1) = P''_i(0) \Rightarrow 2c_{i-1} + 6d_{i-1} = 2c_i \tag{3}$$

where Y_i are also unknowns. However, using (1) and (2), for each subdomain it is straightforward to verify that

$$\begin{cases} a_i = y_i \quad (= P_i(0)) \\ b_i = Y_i \quad (= P'_i(0)) \\ c_i = 3(y_{i+1} - y_i) - 2Y_i - Y_{i+1} \quad (= P''_i(0)/2) \\ d_i = 2(y_i - y_{i+1}) + Y_i + Y_{i+1} \quad (= P'''_i/6) \end{cases} \tag{4}$$

so the Y_i may be used as basic variables and Eq. (3) results for each subdomain in

$$Y_{i-1} + 4Y_i + Y_{i+1} = 3(y_{i+1} - y_{i-1}) \tag{5}$$

Only the first and last Y_i cannot be determined with this set of equations. A usual ("natural") choice to obtain the additional equations is to set $P''_1(0) = 0$ and $P''_N(1) = 0$ in Eq. (3), where N is the number of subdomains. Then the following tridiagonal system of $N+1$ equations is obtained

$$\begin{bmatrix} 2 & 1 & & & \\ 1 & 4 & 1 & & \\ & & \cdots & & \\ & & 1 & 4 & 1 \\ & & & 1 & 2 \end{bmatrix} \begin{bmatrix} Y_1 \\ Y_2 \\ \cdots \\ Y_N \\ Y_{N+1} \end{bmatrix} = \begin{bmatrix} 3(y_2 - y_1) \\ 3(y_3 - y_1) \\ \cdots \\ 3(y_{N+1} - y_{N-1}) \\ 3(y_{N+1} - y_N) \end{bmatrix} \tag{6}$$

which can be efficiently solved using the TDMA (Thomas) algorithm, well known in the CFD literature. Obviously, other boundary conditions may be applied.

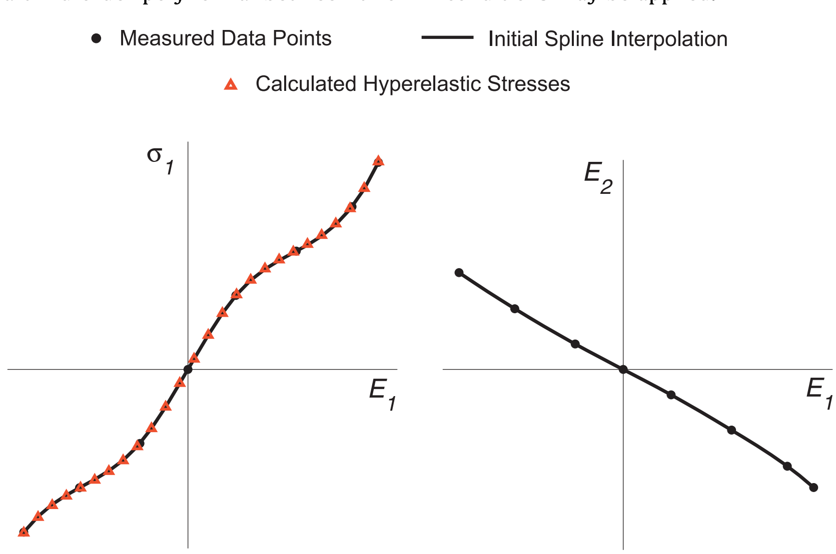

Fig. 1. Assumed "measured" data points $\tilde{\sigma}_1(\widetilde{E}_1)$ and $\widetilde{E}_2(\widetilde{E}_1)$, their initial non-uniform piecewise spline interpolations $\sigma_1(E_1)$ and $E_2(E_1)$ and calculated stresses $\bar{\sigma}_1(\overline{E}_1)$ using Eq. (48) (or Eq. (49)) with the corresponding spline-based energy functions (see Fig. 2).