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A harmonic balance approach for large-scale problems in nonlinear structural dynamics

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ABSTRACT

Harmonic balance (HB) methods allow for rapid computation of time-periodic solutions for nonlinear dynamical systems. We present a filtered high dimensional harmonic balance (HDHB) approach, which operates in the time domain, and provide a framework for implementation into an existing finite element solver. To demonstrate its capabilities, the method is used to solve a set of nonlinear structural dynamics problems related to the field of flapping flight. For each example, the HDHB approach produces accurate steady-state solutions orders of magnitude faster than a traditional time-marching scheme.

1. Introduction

Time-periodic phenomena are frequently encountered in the fields of physics, biology, chemistry, and sociology. Perhaps the most well-known example is vibration, which is ever-present in nature, and is essential to engineering design. In some applications, vibration can be harmful, while in others, it may be desired. The analysis of vibration involves capturing physics in the language of mathematics, resulting in the formulation of a dynamical system.

Natural oscillators are often represented as nonlinear dynamical systems, which in general are too complex to solve analytically, and must be analyzed experimentally or through numerical simulation. This is particularly true for multiphysics problems, which entail simultaneous treatment of various physical phenomena that can occur on a wide range of scales. Fluid–structure interactions serve as an excellent example and are currently gathering substantial interest in industry and academia.

The traditional approach to solving nonlinear dynamical systems begins with a spatial discretization of the governing equations, which is typically accomplished using the finite element method for structures, followed by a temporal discretization based on a finite differencing scheme. Initial conditions are specified for the fully discretized equations of motion, and the solution is incrementally advanced forward in time. This technique is known throughout the literature as time-marching.

The downside to using time-marching methods for time-periodic problems is that they include a transient response in the solution. In many cases, only the steady-state solution is desired. When many degrees of freedom (dof) are involved, the computational time required to achieve a steady-state solution can become excessive.

The harmonic balance (HB) approach is a computationally efficient alternative to time-marching methods for solving time-periodic problems. The approach is based on Fourier analysis, and to its advantage, only includes the steady-state solution. Thus, any computational expense associated with a transient response is completely avoided.

The first formal presentation of the HB method is usually accredited to Kryloff and Bogoliuboff in the 1940s [1]. Throughout the years, many variants of HB technology have emerged. As a result, the technique has been applied to myriad problems in several fields, especially nonlinear circuit analysis [2,3] and nonlinear dynamics [4–8]. A detailed discussion on the many variants of the HB methodology can be found in Dimitriadis' continuation study of higher-order HB solutions [9].

The high dimensional harmonic balance (HDHB) method is a novel modification of the classical HB approach, and was first presented by Hall et al. [10,11]. Instead of working in the Fourier domain, as with the classical HB approach, the HDHB method casts the problem into the time domain. Rather than solving for Fourier coefficients directly, the field variables are discretized in time for one period of oscillation and are solved for numerically. This modification allows for straightforward implementation of the HB methodology into large-scale computational fluid (CFD) and

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structural dynamic (CSD) codes, which already contain standard time-discretization schemes.

One shortcoming of using the HDHB approach for solving nonlinear systems is that it has a tendency to produce nonphysical solutions in addition to the physically meaningful ones that are sought. This effect is known as aliasing. In a comparison of the HB and HDHB approaches for a Duffing oscillator, Liu et al. [12] determined that aliasing terms arise due to the treatment of the nonlinearities in the governing equations. An aliased solution can be identified by a lack of convergence in the Fourier series and can lead to numerical instability.

LaBryer and Attar recently proposed a framework for dealiasing the HDHB approach through judicious filtering of the field variables [13]. They demonstrated that the aliasing terms can be completely removed, or drastically reduced, by filtering in the frequency domain. The drawback to frequency filtering is that repetitive coordinate transformations are required, increasing computational expense. As an alternative to frequency-domain filtering, they showed that filtering in the time domain can produce similar results while reducing the computational burden.

In this paper, we will present the filtered HDHB methodology and demonstrate how it can easily be implemented into an existing finite element solver. The authors have successfully incorporated the technique into an in-house finite element code named ATFEM. To the authors' knowledge, this marks the first application of the HDHB approach to finite element software.

To showcase the capabilities of the HDHB finite element solver, we will present solutions for a suite of nonlinear dynamical problems that relate to the field of flapping flight. First, we investigate the response of a plunging string, which is approximated to be a one-dimensional continuum. Studied next is the motion of a flapping dragonfly wing using two-dimensional plate elements. Finally, we investigate the stresses in a three-dimensional continuum airfoil undergoing forced oscillations.

2. Numerical method

The solution technique presented here is valid for any nonlinear time-periodic system governed by second order partial differential equations (PDEs). PDEs of a different order can be treated in a similar manner. Following a spatial discretization, which can be done using the finite element method for structures, the general form of the resulting ordinary differential equation (ODE) is

$$\mathbf{M}\ddot{\mathbf{u}} + \mathbf{C}\dot{\mathbf{u}} = \mathbf{f}(\mathbf{u}, t), \tag{1}$$

along with the associated initial conditions, where $\mathbf{u}(t)$ is the displacement vector, t is time, and $\mathbf{f}$ is a vector containing the external forces and internal restoring forces, which may be nonlinear. In structural dynamics problems, the matrices $\mathbf{M}$ and $\mathbf{C}$ typically represent mass and damping.

The basic concept of harmonic balancing is to find time-periodic solutions for $\mathbf{u}(t)$. The method proceeds by assuming that the solution to Eq. (1) is smooth and periodic in time with period $T = 2\pi/\omega$, where ω is the fundamental frequency. Consequently, the field variables can be expanded in a Fourier series:

$$\mathbf{u}(t) = \hat{\mathbf{u}}^0 + \sum_{k=1}^{N_H}\left[\hat{\mathbf{u}}^{2k-1}\cos(k\omega t) + \hat{\mathbf{u}}^{2k}\sin(k\omega t)\right], \tag{2}$$

$$\mathbf{f}(t) = \hat{\mathbf{f}}^0 + \sum_{k=1}^{N_H}\left[\hat{\mathbf{f}}^{2k-1}\cos(k\omega t) + \hat{\mathbf{f}}^{2k}\sin(k\omega t)\right], \tag{3}$$

where k is the wavenumber and N_H is the number of harmonics retained in the expansion. The total number of terms in each Fourier series is $N_T = 2N_H + 1$. The only task that remains is to solve for the Fourier coefficients, which can be done using the HB method.

If the smallest timescale associated with the periodic solution of Eq. (1) is given by τ, and N_H is chosen such that $\frac{2\pi}{N_H\omega} \approx \tau$, then the resolution error (or harmonic approximation error) is zero. The simulation can then be considered (in CFD terminology) a direct numerical simulation (DNS). In practice, the timescale τ is unknown, and harmonic approximation error is present. Additional errors associated with approximating the PDE with a finite number of dof (projection error) and the inexact treatment of spatial derivatives (numerical error) are also present.

2.1. Classical harmonic balance method

The classical HB method provides a way to determine the Fourier coefficients in Eqs. (2) and (3). The Fourier series are substituted into the governing equation of motion (1) and a Galerkin projection is performed with respect to the Fourier modes. If we assume (for ease of derivation) that the mass and damping matrices are diagonal, we can write the following system of equations for the Fourier coefficients:

$$\omega^2\mathbf{A}^2\widehat{\mathbf{Q}}\mathbf{M} + \omega\mathbf{A}\widehat{\mathbf{Q}}\mathbf{C} - \widehat{\mathbf{F}} = \mathbf{0} \tag{4}$$

with

$$\widehat{\mathbf{Q}} = \begin{bmatrix} \hat{u}_1^0 & \cdots & \hat{u}_{N_V}^0 \\ \vdots & \hat{u}_i^n & \vdots \\ \hat{u}_1^{2N_H} & \cdots & \hat{u}_{N_V}^{2N_H} \end{bmatrix}, \quad \widehat{\mathbf{F}} = \begin{bmatrix} \hat{f}_1^0 & \cdots & \hat{f}_{N_V}^0 \\ \vdots & \hat{f}_i^n & \vdots \\ \hat{f}_1^{2N_H} & \cdots & \hat{f}_{N_V}^{2N_H} \end{bmatrix},$$

$$\mathbf{A} = \begin{bmatrix} 0 & & & \\ & \mathbf{J}_1 & & \\ & & \ddots & \\ & & & \mathbf{J}_{N_V} \end{bmatrix}, \quad \mathbf{J}_k = \begin{bmatrix} 0 & k \\ -k & 0 \end{bmatrix},$$

where the subscript denotes the dof number, the superscript denotes the Fourier mode, N_V is the total number of dof, and N_H is the total number of Fourier modes. The HB solution array $\widehat{\mathbf{Q}}$ contains $N_T N_V$ terms, which represent Fourier coefficients for displacement at each dof. If the restoring force contained in $\widehat{\mathbf{F}}$ is linear, solutions for the Fourier coefficients can be found analytically [12]. For the nonlinear case, solutions must be obtained numerically. As an alternative, solutions for the harmonic amplitudes A_i^k can be found, along with the peak amplitudes A_i, where

$$A_i^0 = \hat{u}_i^0, \quad A_i^k = \sqrt{\left(\hat{u}_i^{2k-1}\right)^2 + \left(\hat{u}_i^{2k}\right)^2}, \quad A_i = \sum_{k=0}^{N_H} A_i^k. \tag{5}$$

The classical HB method generates $N_T N_V$ analytical expressions to solve for the harmonic amplitudes A_i^k. Implementing this approach for large-scale nonlinear dynamical systems can become cumbersome, especially when many harmonics are retained. Expressions for the nonlinear terms must be developed as a function of the Fourier coefficients, which are often complicated to derive, and do not exist in standard time-marching codes. In order to circumvent this difficulty, a novel extension of the classical method will now be outlined.

2.2. High dimensional harmonic balance method

Hall et al. [10,11] first presented the HDHB method within the context of CFD. The basis of the approach is that instead of working in the Fourier domain, the problem is cast into the time domain. This modification allows for straightforward implementation into large-scale CFD and CSD codes. Expressions for the nonlinear terms in the governing equations do not need to be rederived with respect to the Fourier coefficients. Instead, the Fourier coefficients are related to time domain variables through a discrete Fourier