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Geometrically non-linear force method for assemblies with infinitesimal mechanisms

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Abstract

This paper proposes a force method based structural analysis algorithm for geometrically non-linear bar assemblies. Like the linear force method, the method is suitable for both statically and kinematically indeterminate assemblies, which have been widely used in non-traditional structures such as tension structures and tensegrities. The geometrically non-linear force method is able to produce much more accurate solutions when the infinitesimal mechanisms exist in the assembly, which has been illustrated by a number of examples.

Keywords: Force method; Geometrically non-linearity; Singular value decomposition; Null space basis; Newton–Raphson method; Infinitesimal mechanisms

1. Introduction

Computational structural analysis is normally carried out using displacement-based methods such as the Finite Element Method (FEM) where the displacements are used as variables. It is very effective and well tested. The Force Method (FM) has been more neglected. In comparison with the displacement-based method, the FM has its inherited advantages: the physical concepts of each item in the equations are fairly explicit; and the essential structural and kinematic properties, e.g., the rigid body motions or infinitesimal mobilities, states of the self-stress, etc., can be interpreted clearly from the orthogonal subspace of the equilibrium matrix. These advantages have made the FM an important part of the curriculum in the subject of the structural mechanics. However, the FM is less convenient in terms of the matrix operation. The computation requires a large amount of computer memory and hence becomes computationally expensive. Although some early books on the methods of numerical structural analysis still introduce the FM [1], the frequency of its appearance has steadily declined despite many scholars' effort to revive the FM [2–4]. But this trend has stopped when a family of unorthodox structures, e.g., tension structures and tensegrities, were encountered in 1990s.

For tension structures and tensegrities, the stiffness matrix may not have full rank. The displacement-based methods, which are the most efficient for statically determinate or redundant assemblies, becomes invalid. Although this problem can be dealt with numerically, to employ the FM provides a more satisfactory solution. On the other hand, in the design of tension structures or tensegrities, it is important to know the existence of the states of self-stress and the accompanied finite or infinitesimal mobilities. The displacement-based methods usually do not give such information. It is therefore necessary to seek a different approach.

One of the important papers in the revival of the FM is given by Pellegrino and Calladine [5]. The analysis of the equilibrium matrix was introduced to determine the kinematic determinacy and states of the self-stress. Pellegrino then extended the method to analyze the prestressed

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Nomenclature

$\mathbf{A} = \mathbf{A}_l + \mathbf{A}_{nl}$ the equilibrium matrix ($n_r \times n_c$), where $\mathbf{A}_l$ is the linear part and $\mathbf{A}_{nl}$ is the non-linear part
$\mathbf{B}$ the compatibility matrix ($n_c \times n_r$)
$\mathbf{d} = (\mathrm{d}x \quad \mathrm{d}y \quad \mathrm{d}z)^{\mathrm{T}}$ the nodal displacement vector
$\mathbf{d}'$ the special solution of nodal displacement vector obtained from the compatibility equations
$\mathbf{e}$ the strain vector of the links
$\mathbf{e}_0$ the initial strain vector of the bar
E the elastic modulus
$\mathbf{F}$ the flexibility matrix ($n_r \times n_r$)
$l_{ij} = \sqrt{x_{ij}^2 + y_{ij}^2 + z_{ij}^2}$ the initial length of the bar $i - j$, where $x_{ij} = x_j - x_i$, $y_{ij} = y_j - y_i$, $z_{ij} = z_j - z_i$
$m = n_r - r$ number of inextensional mechanisms
n_c the number of columns in $\mathbf{A}$
n_r the number of rows in $\mathbf{A}$
$\mathbf{q}$ the nodal load vector
r the rank of $\mathbf{A}$
$\mathbf{S} = \mathrm{diag}\{s_{11}, s_{22}, \ldots, s_{rr}\}$ where s_{ii} is the singular value
$s = n_c - r$ number of states of self-stress
$\mathbf{t}$ the internal axial force vector
$\mathbf{t}'$ the special solution of internal axial force vector obtained from the equilibrium equations
$\mathbf{U} = [\mathbf{U}_r \quad \mathbf{U}_m]$ the unitary orthogonal matrix, here $\mathbf{U}_r$ is the left singular vectors and $\mathbf{U}_m$ is modes of inextensional mechanisms respectively
$\mathbf{V} = [\mathbf{V}_r \quad \mathbf{V}_s]$ the unitary orthogonal matrix, where $\mathbf{V}_r$ is the left singular vectors and $\mathbf{V}_s$ is independent modes of self-stress respectively
W the external forces applied at node
$\mathbf{X}_l = (x_l \quad y_l \quad z_l)^{\mathrm{T}}$ the Cartesian coordinates of node l in its original positions
α vector containing s real numbers
β vector containing m real numbers
$\Omega_{\mathbf{d}}, \Omega_{\mathbf{q}}$ convergence parameter based on residual displacements and residual forces respectively
$(\,)^p$ denote the corresponding response at step (p)

structural assemblies such as the cable networks [6]. However, only linearity was considered in his analysis though the infinitesimal mobilities, if they existed, could lead to geometrical non-linearity. A more precise non-linear analysis is therefore required to produce more accurate results of displacements. This paper deals with this matter.

The layout of the paper is as follows. In Section 2, we introduce the basic principle of the FM based on the matrix analysis. It is followed by the geometrically non-linear analysis procedures in Section 3. A few examples are given in Section 4, some of which were first included in [6] using linear analysis. The results are compared. Section 5 concludes the paper with discussion and some suggestions for future work.

2. The matrix force method

The FM is best illustrated using a bar assembly. Its equilibrium and compatibility equations can be written in the following matrix forms:

$$\mathbf{At} = \mathbf{q} \tag{1}$$

$$\mathbf{Bd} = \mathbf{e} \tag{2}$$

It can be proved that [8]

$$\mathbf{B} = \mathbf{A}^{\mathrm{T}}$$

Thus

$$\mathbf{A}^{\mathrm{T}}\mathbf{d} = \mathbf{e} \tag{3}$$

Assume that the material remains linear elastic. We have

$$\mathbf{e} = \mathbf{e}^0 + \mathbf{Ft} \tag{4}$$

The solution can normally be obtained from Eqs. (1), (2) and (4) unless the assembly is statically or kinematically indeterminate in which $\mathbf{A}$ become a rectangular matrix. In this circumstance, the solution can only be calculated through the null space basis and subspace of the equilibrium matrix $\mathbf{A}$, which is explained next.

Assume that the dimension of $\mathbf{A}$ is $n_r \times n_c$. The SVD expression of $\mathbf{A}$ is

$$\mathbf{A} = \mathbf{U}\begin{bmatrix} \mathbf{S} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix}\mathbf{V}^{\mathrm{T}} \tag{5}$$

Substituting Eq. (5) into Eqs. (1) and (3) gives

$$\mathbf{t} = \mathbf{t}' + \mathbf{V}_s\alpha \tag{6}$$

and

$$\mathbf{d} = \mathbf{d}' + \mathbf{U}_m\beta \tag{7}$$

$\mathbf{t}'$ and $\mathbf{d}'$ are the special solution of Eqs. (1) and (3) and can be expressed as

$$\mathbf{t}' = \mathbf{V}_r\mathbf{S}^{-1}\mathbf{U}_r^{\mathrm{T}}\mathbf{q} \tag{8}$$

$$\mathbf{d}' = \mathbf{U}_r\mathbf{S}^{-1}\mathbf{V}_r^{\mathrm{T}}\mathbf{e} \tag{9}$$

Substituting Eq. (6) into Eq. (4) yields

$$\mathbf{e} = \mathbf{e}^0 + \mathbf{Ft} = \mathbf{e}^0 + \mathbf{F}(\mathbf{t}' + \mathbf{V}_s\alpha) \tag{10}$$

Because [7]

$$\mathbf{V}_s^{\mathrm{T}}\mathbf{e} = \mathbf{0} \tag{11}$$

Eq. (10) becomes

$$\mathbf{V}_s^{\mathrm{T}}(\mathbf{e}^0 + \mathbf{Ft}') + \mathbf{V}_s^{\mathrm{T}}\mathbf{FV}_s\alpha = \mathbf{0} \tag{12}$$

from which α can be deduced.

β of Eq. (7) can be determined according to the virtual work principle