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Elastoplastic analysis of plane stress/strain structures via restricted basis linear programming

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ABSTRACT

In this paper, elastoplastic analyses of plane-stress and plane-strain structures are addressed. The traditional von Mises yield surface is assumed in the material constitutive model and its piecewise-linear (PWL) approximation is derived. An associated flow rule is adopted and a consistent linear mixed hardening rule is developed in the so called mixed formulation. The relevant mathematical programming (MP) problem, which aims at the maximization of the linear combination of plastic multipliers, is constructed. Restricted basis linear programming (RBLP) is used to solve the MP problem, while special provisions are accounted for to obtain nonholonomic solutions. In order to reduce the required storage space, the Revised Simplex method and a sifting technique are employed. The proposed algorithm is validated and its performance is illustrated by solving some classical nonlinear problems.

1. Introduction

Analysis of structures composed of elastoplastic materials is still a growing area in the field of structural mechanics. The need for realistic responses of structures to the increasing loads (such as heavy live loads, crash loads, lateral quake/wind loads, and explosions) or need for safety assessment of structures (commonly required in limit load analysis and design procedures), have drawn attention of several researchers toward this subject [1–3].

Using piecewise-linear (PWL) yield surfaces in combination with optimization techniques has opened a new horizon of study known as PWL elastoplasticity [4]. Mathematical programming (MP) has been employed by several researchers in nonlinear analysis of structures and has found to be a very robust and versatile approach to solution of problems of this kind. MP based approaches do not contain the difficulties of the iterative approaches such as the implementation of cumbersome return algorithms, the enforcing of convergence criteria, and so on. Satisfying the yield and equilibrium equations at global scale, unconditional stability (that appears in step-by-step solution schemes) and use of easily coded optimization algorithms (frequently available in commercial software) are encouraging features of such approaches that motivate researches in this area. In the following, the history of such approaches is cited briefly.

Mathematical linear programming has been considered for rigid-plastic limit analysis (LA) of framed structures and its historical and theoretical background has been deeply discussed and demonstrated in many nonlinear analysis text books, e.g. see [5]. Maier [6,7] proposed the use of quadratic programming (QP) in elastoplastic analyses and derived a matrix formulation for framed structures governed by PWL constitutive models [8]. The use of linear complementarity problem (LCP) solvers was also found to be efficient and a restricted basis linear programming (RBLP) was proposed as an alternative to QP [9]. Therefore LCP concept was extended to various engineering problems such as dynamic analysis [10], shakedown analysis [11], and softening frames [12,13]. Also some researchers dealt with piecewise-linearization of yield surfaces, so as to be utilized in optimization approaches as linear constraints [14–16]. Recently a modified version of RBLP has been proposed in the spirit of framed structures, which automatically captures and handles any local unloading and removes any need for sub-problem solution in the cases of reaching yield surface corners [17,18]. This approach has been successfully used in elastoplastic analysis of softening frames and the proposed maximization criterion has shown an excellent capability in capturing the exact response of structures. This approach, which basically works in an incremental manner, preserves the distinct features of the step-by-step method, namely exactness and unconditional stability, while removes its disadvantages addressed in [12].

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In spite of pervasive studies on MP approaches applied to skeletal structure analyses, which have made this topic a well-developed area, it has been rarely utilized in direct analysis of plane stress/strain problems. Kaliszky and Lógó [19] presented a mixed variational principle for plane-strain problems characterized by bi-linear hardening materials. In this approach the load multiplier is maximized using nonlinear MP solvers in the view of nonlinear nature of constraints. Another development in MP approaches toward 2D-stress/strain problems has appeared in [20], where traditional Mohr–Coulomb yield surface is piecewise-linearized and used for safety assessment by load factor maximization. Utilizing this approach, which is efficiently improved by the aid of sifting and re-meshing techniques, the holonomic response of structures and corresponding limit load are detected with a reasonable accuracy.

The most recent contribution to direct elastoplastic analysis of plane stress and plane strain structures by the aid of optimization tools is the complementarity approach proposed by Tangaramvong et al. [21]. This approach implements a mixed finite element formulation, developed by Capsoni and Corradi for quadrilateral bilinear elements [22,23], in constructing an MP problem. As for constraints, the von Mises or Tresca yield criteria are considered in their original nonlinear forms. The resulted MP is solved using an industry-standard complementarity solver GAMS/PATH with an interface for MATLAB environment. This approach belongs to holonomic solution category and sufficiently small load steps are needed to reduce the amount of errors appearing due to possible local unloadings. In this approach a relatively large fraction of the CPU time (27–82% in some studied structures) is spent for load estimations beyond the limit load, i.e. infeasible load steps.

In this paper, the RBLP is extended to 2D-Stress/Strain problems following a similar approach discussed in [17,18]. To this end, theoretical aspects of problem including: field approximation, piecewise-linearization of von Mises yield model in 2D-stress/strain, and development of linear mixed hardening constitutive laws are presented in Section 2. Formulation of the problem and its implementation in the MP problem are explained in Section 3. In Section 4, the solution procedure of the mathematical programming problem is discussed and finally, in Section 5, some numerical examples are presented to demonstrate the capabilities of the proposed algorithm and numerically validate its results.

Bold-face, regular, and italic symbols are adopted herein for matrices, vectors and scalars, respectively. Superscript T means transpose, and dots stand for rates (i.e. derivative with respect to ordering, not necessarily physical, time).

2. Theoretical formulation

2.1. Field approximations

It is well-known that in LA by popular FEM the computed safety factor might severely be affected by locking, see e.g. [24]. Herein a multi-field mixed discretization is adopted: the pairs of conjugate variables are modeled and the conservation of the scalar product for conjugate fields is imposed in a weak form. In such approaches, the discrete problem is formulated in terms of generalized variables and using appropriate shape functions, which lets to rule out the shear locking phenomena by relaxing the kinematic constraints that induce locking. The theoretical aspects of mixed formulation is deeply studied in literature, see e.g. [25–27], and are not brought here for the sake of brevity. In this study, quadrilateral elements are considered and the displacement field, $u_e(\xi,\eta)$, within the element e is approximated by quadratic isoparametric shape functions, $\mathbf{N}$, in the space of natural coordinates (ξ,η), and with reference to the nodal displacements U_e:

$$u_e(\xi,\eta) = \mathbf{N}U_e \tag{1}$$

Four Gauss points, $g = 1$–4, over each element are used for numerical integration and the stress and strain fields are assumed to vary linearly over the element. Herein the actual stress components corresponding to the Gauss points are assumed as the element generalized stress, $\bar{\sigma}_e$, and bilinear shape functions, $\mathbf{\Psi}^e_\sigma$, which are referring to the Gauss points instead of element nodes, are used to interpolate stresses, $\sigma_e(\xi,\eta) = [\,\sigma_x(\xi,\eta),\quad \sigma_y(\xi,\eta),\quad \tau_{xy}(\xi,\eta)]^T_e$, over the element:

$$\sigma_e(\xi,\eta) = \mathbf{\Psi}^e_\sigma \bar{\sigma}_e \tag{2}$$

Note that the (3×12) stress shape function matrix, once computed at the gth Gauss point, collects three zero blocks and an identity matrix $\mathbf{I}_{3\times 3}$, located at the block corresponding to the considered Gauss point, e.g. for $g = 1$:

$$\mathbf{\Psi}^e_\sigma\big|_{g=1} = [\,\mathbf{I}_{3\times 3}\quad \mathbf{0}_{3\times 3}\quad \mathbf{0}_{3\times 3}\quad \mathbf{0}_{3\times 3}\,] \tag{3}$$

Also the strain field $\varepsilon_e(\xi,\eta) = [\,\varepsilon_x(\xi,\eta),\quad \varepsilon_y(\xi,\eta),\quad \gamma_{xy}(\xi,\eta)]^T_e$, within the element, is approximated by the shape functions $\mathbf{\Psi}^e_\varepsilon$ and governed by the element generalized strain vector $\bar{\varepsilon}_e$ as follows:

$$\varepsilon_e(\xi,\eta) = \mathbf{\Psi}^e_\varepsilon \bar{\varepsilon}_e \tag{4}$$

In order to preserve the scalar product of conjugate quantities in terms of actual model variables and the generalized ones, the strain field shape function over the element domain Ω is selected as follows:

$$\mathbf{\Psi}^e_\varepsilon = \mathbf{\Psi}^e_\sigma \left(\int_\Omega \mathbf{\Psi}^{e^T}_\sigma \mathbf{\Psi}^e_\sigma d\Omega\right)^{-1} \tag{5}$$

Also this shape function (3×12) matrix, once computed at the gth Gauss point, collects three zero blocks and a scaled identity matrix. The scale factor is $(tJ_g)^{-1}$ wherein t is the element thickness and J_g is the Jacobian matrix determinant, calculated at the Gauss point g. For instance, at the first Gauss point ($g = 1$) the strain shape function matrix becomes:

$$\mathbf{\Psi}^e_\varepsilon\big|_{g=1} = \left[\frac{1}{tJ_{g=1}}\mathbf{I}_{3\times 3}\quad \mathbf{0}_{3\times 3}\quad \mathbf{0}_{3\times 3}\quad \mathbf{0}_{3\times 3}\right] \tag{6}$$

By weighting the strain definition relation, the consistency matrix $\mathbf{C}_e$ for linear kinematics is determined as:

$$\mathbf{C}_e = \int_\Omega \mathbf{\Psi}^{e^T}_\varepsilon \nabla \mathbf{N} d\Omega \tag{7}$$

wherein, ∇ is the well-known symmetric gradient operator, generating the local strain field from the displacement field. The consistency matrix $\mathbf{C}_e$, the nodal displacements U_e and the nodal forces f_e can be related to the generalized stresses and strains by the following relations:

$$\begin{aligned} \bar{\varepsilon}_e &= \mathbf{C}_e U_e \\ f_e &= \mathbf{C}^T_e \bar{\sigma}_e \end{aligned} \tag{8}$$

Accordingly, the element stiffness matrix reads:

$$\mathbf{K}_e = \mathbf{C}^T_e \overline{\mathbf{D}}_e \mathbf{C}_e \tag{9}$$

wherein the generalized material stiffness matrix $\overline{\mathbf{D}}_e$ is evaluated in terms of material constitutive stiffness matrix $\mathbf{D}_e$ and strain shape functions through the following integral:

$$\overline{\mathbf{D}}_e = \int_\Omega \mathbf{\Psi}^{e^T}_\varepsilon \mathbf{D}_e \mathbf{\Psi}^e_\varepsilon d\Omega \tag{10}$$