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A space-averaged model of branched structures

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ABSTRACT

Many biological systems and artificial structures are ramified, and present a high geometric complexity. In this work, we propose a space-averaged model of branched systems for conservation laws. From a one-dimensional description of the system, we show that the space-averaged problem is also one-dimensional, represented by characteristic curves, defined as streamlines of the space-averaged branch directions. The geometric complexity is then captured firstly by the characteristic curves, and secondly by an additional forcing term in the equations. This model is then applied to mass balance in a pipe network and momentum balance in a tree under wind loading.

1. Introduction

Branched systems are ubiquitous in nature and man-made structures. In biological systems, ramification is a mean for increasing exchange surfaces at a given mass; this is commonly observed in blood circulation, pulmonary system [1], and plants like trees and bushes [2], to list a few. In these various systems, an accurate modeling of fundamental conservation laws is of crucial importance, be it for medical purposes, ecological applications or predictions of mechanical failure.

Over the past decades, numerous studies helped uncover the flow kinematics in the blood system, for instance for targeted drug delivery [3], or in lungs for finding geometries that maximize ventilation in a limited time [4,5] and to study the behavior of liquid plugs [6,7]. In plants, various studies have been designed to understand the static and dynamic response to external flows [8–10]. The modeling complexity of such systems comes from the multiple ramifications and branching points. In these branched systems, robust models exist for individual segments, but branched systems are not easily modeled and often require heavy computations. A key issue is to find a continuous way for modeling these geometries.

A typical example is that of trees submitted to external flows. A continuous representation of a tree as a tapered beam was proposed by McMahon for analyzing the mechanical stability of a tree under its own weight [11]. This model captures efficiently some key geometric features of tree-like structures and allow for computing accurately the wind-induced loads on an isolated plant [10,12]. However, this continuous approach does not account for the changes in branch orientation, and the tree effect on the flow cannot be modeled inside the tree crown. To overcome this issue, many models are based on fractal models for trees [13–15]. Such models rely on costly computations and a large number of parameters. Moreover, despite the variety of existing models, there is a lack of a general formulation of conservation laws in branched systems.

In this paper, we present a new model for space-averaged branching (SAB) in conservation laws. The purpose of this work is to provide a continuous formulation of conservation laws in branched systems, represented by a small number of parameters and applicable to a large variety of problems, in particular for solving full fluid–structure computations on branched systems through a porous medium approach. More specifically, we expect that the proposed approach will help in modeling complex structures involving large number of branching, avoiding the fine description of each and every segment. The present SAB model is inspired from homogenization techniques and porous media approach. We obtain an equivalent problem where a branched system is represented by independent characteristic curves, on which specific conservation equations are solved. These characteristic curves correspond to streamlines of the average branch direction, as sketched in Fig. 1. The SAB model is derived in Section 2. We present then two applications of the model, first on a case study of flow rate computation in a simple pipe network in Section 3, and then on the problem of trees submitted to an external flow in Section 4. Finally, a general discussion and conclusion is given in Section 5.

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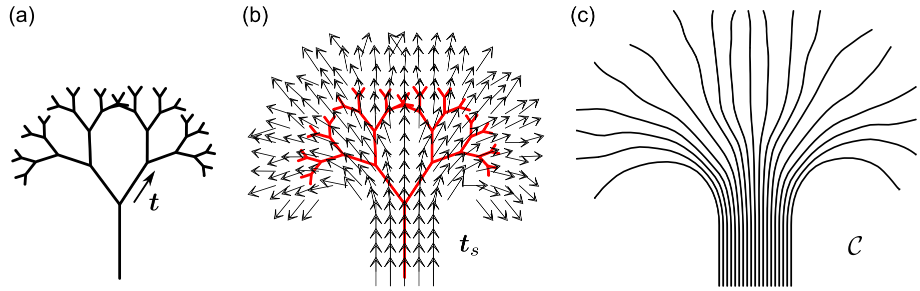

Fig. 1. Space-averaged branching model: (a) ramified system with oriented branches ($\boldsymbol{t}$), (b) volume averaged branch direction $\boldsymbol{t}_s$, and (c) characteristic curves $\mathcal{C}$ equivalent to the branched system in SAB model.

2. Space-averaged branching model

2.1. Definitions and problem equations

We consider a branched system where the segments between two branching points are oriented and described by the corresponding tangent vector $\boldsymbol{t}$ (Fig. 1a). The segments are slender, so that a segment length is much larger than its transverse dimension. The resulting description is thus one-dimensional along the segments. Under these assumptions, a general formulation of the conservation of a vectorial quantity $\boldsymbol{Q}$ along the system is

$$\frac{\mathrm{d}\boldsymbol{Q}}{\mathrm{d}x} + \boldsymbol{G}(x) = 0, \quad \sum \boldsymbol{Q}^{-} = \sum \boldsymbol{Q}^{+}, \tag{1}$$

where $\boldsymbol{G}$ is a forcing term, x the curvilinear coordinate, and we use superscript $-$ (resp. $+$) to characterize a segment oriented towards (resp. away from) the branching point, according to the system orientation given by the tangent vector $\boldsymbol{t}$ (Fig. 2). The first relation is the conservation of $\boldsymbol{Q}$ along a segment, and the second one gives a relation at the nodes of the structure. Such conservation equations are ubiquitous in branched systems, and their complexity arises from the discontinuities introduced by branching nodes. In the general case, Eq. (1) is a vectorial equation, but can be decomposed into a set of scalar equations by projection on a fixed frame. In the following, we derive the SAB model considering a scalar problem, $\mathrm{d}Q/\mathrm{d}x + G(x) = 0$.

Whereas the initial branched system has no volume (1D description), it is necessary to introduce its finite volume for averaging purposes (see Fig. 2a). In order to obtain space-averaged quantities, we introduce a representative volume Ω and denote Ω_s the volume occupied by the branched system included in Ω, as sketched in Fig. 2a. We define the volume fraction $\varphi = \Omega_s/\Omega$. The representative volume Ω must be large compared to the typical diameter of the branched system's segments [16]. We use a standard space average operator over the branched system, noted $\langle\cdot\rangle_s$,

$$\langle\cdot\rangle_s = \frac{1}{\Omega_s}\int_{\Omega_s} \cdot\,\mathrm{d}\Omega, \tag{2}$$

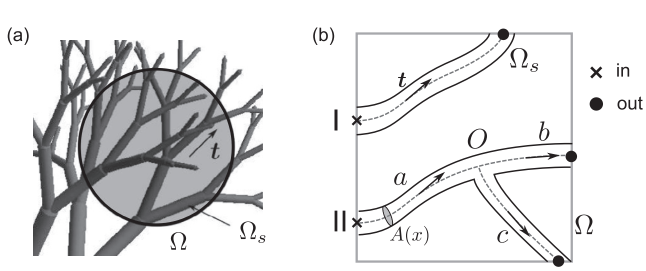

Fig. 2. (a) Space averaging domain and (b) example for the averaging method and corresponding notations.

for a quantity Q defined in the system. This formalism is typically used in porous media analysis, where Ω_s stands for the volume occupied by a solid and $\Omega - \Omega_s$ is occupied by a fluid [17,18].

2.2. Volume equation derivation

For any quantity $Q(x)$, where x is the curvilinear coordinate along the segment, we introduce in the volume Ω_s a continuously differentiable function q corresponding to Q per unit section, so that in a cross section normal to $\boldsymbol{t}$,

$$q = q(x) = \frac{Q(x)}{A(x)}, \tag{3}$$

where $A(x)$ is the local cross-section. This definition yields some singularities at the branching points and at the borders of the averaging volume. Due to the high slenderness of the segments, these singularities are easily overcome without loss of generality; these technical points are discussed in A.

We consider the sketch of Fig. 2b for obtaining volume equations. We denote Q^{in} (resp. Q^{out}) the sum of Q where the segments go into Ω (resp. out of Ω) with respect to $\boldsymbol{t}$. According to the previous notations and slenderness hypothesis, we can write for segment I in Fig. 2b

$$Q_I^{\mathrm{out}} - Q_I^{\mathrm{in}} = \oint_{\partial\Omega_{sI}} \frac{Q}{A}\boldsymbol{t}\cdot\boldsymbol{n}_{\Omega_s}\,\mathrm{d}\mathcal{S} = \oint_{\partial\Omega_{sI}} q\boldsymbol{t}\cdot\boldsymbol{n}_{\Omega_s}\,\mathrm{d}\mathcal{S}, \tag{4}$$

where $\partial\Omega_{sI}$ is the border of segment I. Since q is continuously differentiable in Ω_s, we can apply the divergence theorem. We can then introduce the space-average operator as defined in Eq. (2), and we use a special property for the volume average of the spatial divergence, noted ∇,

$$\varphi\langle\nabla\cdot q\boldsymbol{t}\rangle_s = \nabla\cdot(\varphi\langle q\boldsymbol{t}\rangle_s) + \frac{\varphi}{\Omega_s}\int_{\partial\Omega_s} q\boldsymbol{t}\cdot\boldsymbol{n}\,\mathrm{d}\mathcal{S}, \tag{5}$$

where $\partial\Omega_s$ the interface between Ω and Ω_s, and $\boldsymbol{n}$ the normal to the interface oriented towards Ω_s [16]. As a result, the sum over each independent segment (here noted I and II) gives

$$Q^{\mathrm{out}} - Q^{\mathrm{in}} = \frac{\Omega_s}{\varphi}\nabla\cdot(\varphi\langle q\boldsymbol{t}\rangle_s). \tag{6}$$

We consider now the conservation equations given in Eq. (1), which give, for segment I (or II) of Fig. 2b,

$$Q_I^{\mathrm{out}} - Q_I^{\mathrm{in}} = -\int_{\mathrm{in}_I}^{\mathrm{out}_I} G\,\mathrm{d}x = -\int_{\Omega_{sI}} g\,\mathrm{d}\Omega. \tag{7}$$

The same analysis can be done on segment II using the conservation of Q at a branching point, leading in the general case to

$$Q^{\mathrm{out}} - Q^{\mathrm{in}} = -\int_{\mathrm{in}}^{\mathrm{out}} G\,\mathrm{d}x, \tag{8}$$