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Improved semi-analytical sensitivity analysis using a secant stiffness matrix for geometric nonlinear shape optimization

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ABSTRACT

This work presents a semi-analytical sensitivity analysis approach for geometric nonlinear shape optimization. A secant stiffness matrix is used in the nonlinear solution procedure. Conditions that an accurate derivative of the matrix should satisfy are determined. Following these conditions, a correction term for the finite differencing approximation is constructed. Due to the asymmetry of the secant stiffness matrix, the correction term is expressed in the product spaces of two sets of zero eigenvectors. The analytical formulas of these vectors are also presented, which increases the computational efficiency. Numerical examples highlight the ability of the technique to effectively eliminate sensitivity analysis errors.

1. Introduction

Nonparametric shape optimization takes the nodal coordinates of finite element models as design variables, which provides a much larger design space than parameterized shape optimization. However, the large number of design variables requires methods for highly efficient sensitivity evaluations when a gradient based algorithm is used to solve the optimization problem.

The most common methods to obtain sensitivities are the global finite difference method, the variational method and the discrete method [1–3]. The implementation of the global finite difference method is straightforward. However, the method suffers from errors when the perturbation size is either too large or too small, and, more importantly, it suffers from large computational costs owing to repeated, time-consuming structural analyses. The variational method first differentiates the continuum equation of a structural system, and thereafter discretizes the formula, while the differentiations occur directly on the discretized governing equations of finite element systems for the discrete method. Due to the convenience of a straightforward implementation within finite element codes, the discrete method is considered in this study.

Because the analytical derivatives of discrete quantities such as the stiffness matrix, mass matrix, and load vector, are difficult to obtain, semi-analytical methods, where analytical derivatives are approximated by finite differencing, have been widely employed. The significant accuracy problem related to semi-analytical approaches has been well studied [4–6]. The problem is recognized to be associated with rigid body rotations of the finite elements. A variety of techniques have been presented to eliminate this error, among which exact semi-analytical sensitivity [7,8] and refined semi-analytical analysis [9,10] are able to eliminate the error in linear problems. The exact semi-analytical technique adds a correction term to the finite differencing approximation so that a set of rigid-body conditions are satisfied. The correction term is expressed in the product spaces of a set of zero eigenvectors. The refined semi-analytical technique focuses on correcting the approximation error of the derivatives of internal force vectors, and the correction term is a linear combination of the set of zero eigenvectors. The refined semi-analytical approach has also been extended to geometric nonlinear cases.

In this study, an accurate geometric nonlinear sensitivity analysis method using a secant stiffness matrix is investigated. Firstly, in Section 2, the tangent stiffness matrix and, in particular, the secant stiffness matrix utilized in the proposed nonlinear analysis procedure are introduced. The discrete semi-analytical sensitivity analysis in conjunction with the adjoint approach is described in Section 3. In Section 4, the exact semi-analytical sensitivity analysis method is extended to the nonlinear case, where the form of the correction term is presented. The sensitivity results of both reaction force and stress responses for a cantilever beam problem are presented and discussed in Section 5. Finally the conclusions are presented in Section 6.

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2. Secant and tangent stiffness matrices for nonlinear finite element analysis

2.1. Nonlinear analysis procedure with secant and tangent stiffness matrices

The Newton–Raphson method is often used to solve nonlinear problems iteratively [11]. In the initial step (for $i = 0$), a linear system of equations involving the tangent stiffness matrix $\mathbf{K}_T$ is solved:

$$\mathbf{K}_T(\mathbf{0}, \mathbf{U}^p) \cdot \begin{bmatrix} \mathbf{U}^{f,0} \\ \mathbf{U}^p \end{bmatrix} = \begin{bmatrix} \mathbf{F}^f \\ \mathbf{F}^{p,0} \end{bmatrix} \quad (1)$$

where $\mathbf{U}^p$ and $\mathbf{F}^f$ represent the prescribed displacement vector and the given external force vector respectively, $\mathbf{U}^f$ is the unknown displacements and $\mathbf{F}^p$ represents the unknown reaction force vector that corresponds to $\mathbf{U}^p$. The incremental reaction forces and displacements of the next iteration are obtained by solving the following linear problem:

$$\mathbf{K}_T\left(\mathbf{U}^{f,i}, \mathbf{U}^p\right) \cdot \begin{bmatrix} \Delta\mathbf{U}^{f,i+1} \\ \mathbf{0} \end{bmatrix} = \begin{bmatrix} \mathbf{R}^{f,i} \\ \Delta\mathbf{F}^{p,i+1} \end{bmatrix} \quad (2)$$

Both the unknown displacements and the reaction forces are then updated by

$$\mathbf{U}^{f,i+1} = \mathbf{U}^{f,i} + \Delta\mathbf{U}^{f,i+1} \quad (3)$$

$$\mathbf{F}^{p,i+1} = \mathbf{F}^{p,i} + \Delta\mathbf{F}^{p,i+1} \quad (4)$$

The procedure terminates when the residual force vector is sufficiently small.

In Eq. (2), $\mathbf{R}^f$ is the residual force vector, and it equals to the difference between external force vector and configuration-dependent internal force vector. Two methods are normally used in calculating the internal force vector. One method assembles the internal forces of the individual finite elements. The other method utilizes the so-called secant stiffness matrix. Fig. 1 depicts the force–displacement curve of a general nonlinear problem. The secant stiffness matrix $\mathbf{K}_S(\mathbf{U})$ is the slope of the secant line connecting points on the equilibrium curve with the initial point. The internal force vector

$\mathbf{F}^{int}$ is evaluated through the following equation [12,13]:

$$\mathbf{F}^{int} = \mathbf{K}_S(\mathbf{U}) \cdot \mathbf{U} \quad (5)$$

The use of the secant stiffness matrix has been investigated by various authors, and new solution strategies have been developed for geometric nonlinear problems [14–17]. The secant stiffness matrix is also employed in the analysis limit behavior of structures [16], and the estimation of limit and bifurcation load factors [18,19]. Therefore, the secant stiffness matrix is used in the present study in the calculation of the internal forces due to its potential applications in nonlinear structural analysis problems.

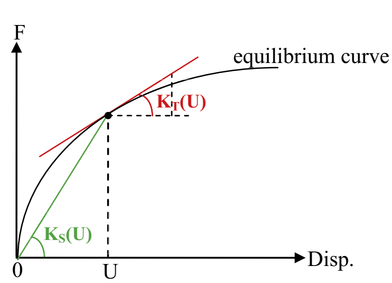

Fig. 1. Secant $\mathbf{K}_S$ and tangent $\mathbf{K}_T$ stiffness matrices at an equilibrium point.

From the standpoint of the secant stiffness matrix, the governing equation of a nonlinear system is

$$\mathbf{K}_S(\mathbf{U}) \cdot \begin{bmatrix} \mathbf{U}^f \\ \mathbf{U}^p \end{bmatrix} = \begin{bmatrix} \mathbf{F}^f \\ \mathbf{F}^p \end{bmatrix} \quad (6)$$

and the nonlinear residual force vector $\mathbf{R}$ is thus expressed as

$$\begin{bmatrix} \mathbf{R}^f \\ \mathbf{R}^p \end{bmatrix} = \mathbf{K}_S(\mathbf{U}) \cdot \begin{bmatrix} \mathbf{U}^f \\ \mathbf{U}^p \end{bmatrix} - \begin{bmatrix} \mathbf{F}^f \\ \mathbf{F}^p \end{bmatrix} \quad (7)$$

2.2. Secant stiffness matrix based on the Green–Lagrange strain

The expression of the secant stiffness matrix has been derived for several types of finite elements by Pedersen [20,21]. The components of the nonlinear Green–Lagrange strain tensor ε_{ij} in Cartesian coordinates are defined by

$$\varepsilon_{ij} = \frac{1}{2}(u_{i,j} + u_{j,i}) + \frac{1}{2}(u_{k,i} \cdot u_{k,j}) \quad (8)$$

where variables u represent a displacement field and i, j and k represent the three directions x, y, and z, respectively, in three-dimensional (3D) space. The Einstein summation convention applies for the index k. In matrix form, this is given as

$$\boldsymbol{\varepsilon} = \mathbf{B}^L\mathbf{U} + \frac{1}{2}\mathbf{U}^T\mathbf{B}^N\mathbf{U} \quad (9)$$

where $\mathbf{B}^L$ is the strain–displacement matrix in the linear problem, and $\mathbf{B}^N$ is a symmetric matrix. With a variation of $\mathbf{U}$, we obtain

$$\delta\boldsymbol{\varepsilon} = \left(\mathbf{B}^L + \mathbf{U}^T\mathbf{B}^N\right)\delta\mathbf{U} \quad (10)$$

The stress measure conjugate to the Green–Lagrange finite strain tensor is the second Piola–Kirchhoff stress tensor $\boldsymbol{\tau}$. The relationship between them is

$$\boldsymbol{\tau} = \mathbf{D}\boldsymbol{\varepsilon} \quad (11)$$

where $\mathbf{D}$ is the material matrix. In addition, the general equilibrium equation due to the principle of virtual work over an element volume is

$$\int \delta\boldsymbol{\varepsilon}^T\boldsymbol{\tau} dV = \delta\mathbf{U} \cdot \mathbf{F} \quad (12)$$

Inserting Eqs. (9)–(11) into Eq. (12), the equilibrium equation is derived as

$$\int \left(\mathbf{B}^L + \mathbf{U}^T\mathbf{B}^N\right)^T \mathbf{D}\left(\mathbf{B}^L + \frac{1}{2}\mathbf{U}^T\mathbf{B}^N\right) dV \cdot \mathbf{U} = \mathbf{F} \quad (13)$$

This results in the secant stiffness matrix:

$$\mathbf{K}_S(\mathbf{U}) = \int \left(\mathbf{B}^L + \mathbf{U}^T\mathbf{B}^N\right)^T \mathbf{D}\left(\mathbf{B}^L + \frac{1}{2}\mathbf{U}^T\mathbf{B}^N\right) dV \quad (14)$$

Note that unlike the tangent stiffness matrix, the secant stiffness matrix described here is asymmetric. Closed form formulas of the secant stiffness matrix for several types of finite elements, including the plane triangular element, the axisymmetric element with triangular cross-section and the 3D 4-node tetrahedral element, are found in [20,21].

3. Semi-analytical sensitivity analysis using the secant stiffness matrix

Because the number of design variables is usually much larger than the number of system responses in nonparametric shape optimization, the adjoint sensitivity analysis approach is preferable for efficiency.