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Efficient and accurate calculation of sensitivity of damped eigensystems

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ABSTRACT

The exact methods (the algebraic method and Nelson's method) to calculate eigensensitivities need a matrix decomposition for each eigensensitivity, and therefore are time-consuming. The modal method may be the most efficient method for eigensensitivity analysis, but suffers from modal truncation error. A method is presented by exactly expressing the modal truncation error as a sum of available modes and system matrices. The proposed method maintains original space without having to use the state-space form. It is shown that the proposed method yields good trade-off between accuracy and computational complexity in terms of the theoretical analysis and numerical studies.

1. Introduction

Structural design is a process to enhance the performance of a mechanical structure or system by changing its design parameters. Sensitivity analysis deals with the calculations of the rate of performance measure from changes in the design parameters of engineering structures. A significant body of research has been devoted to the calculation and application of sensitivity analysis in engineering designs (see, e.g., van Keulen et al. [1], Choi and Kim [2] or Haug et al. [3]). Computational methods of sensitivity analysis have received much attention over the past decades, particularly those related to the eigenvalue problem. It is well-known that frequencies and mode shapes (eigensolutions, modes or eigenpairs) of a structure represent the dynamic characteristics of the structure. Therefore, eigensensitivity analysis plays an integral role in many design methodologies of engineering problems, including structural modal reanalysis [4], dynamic modification [5], optimization [6], reliability [7], model updating [8,9] and structural health monitoring [10]. A significant body of the computation and application of eigensensitivity can be seen in Adelman and Haftka [11,12] or Chen [13]. Although calculating eigenvalue derivative is straightforward, determining eigenvector sensitivity raises several challenges, due in part to the singularity of the coefficient matrix of the linear equation of the eigenvector derivatives (in this paper, we call it *the singularity problem*).

Fox and Kapoor [14] developed a direct algebraic method to obtain the sensitivities of mode shapes. The algebraic method obtains the sensitivities of mode shapes by assembling the derivatives of eigenproblems and the additional constraints obtained from the derivative of normalization into a linear system of algebraic equations. Lee and Jung [15] derived an algebraic method, which assembles a linear system of algebraic equations with symmetric coefficient matrices such that this approach can calculate the sensitivities of mode shapes with little computational cost. Later, Lee et al. [16] further extended their algebraic method to symmetric viscously damped systems. Choi et al. [17], Guedria et al. [18], Chouchane et al. [19] and Xu et al. [20] further developed some algebraic methods to obtain the eigensensitivities of asymmetric viscously damped systems. Recently, Li et al. [21] suggested a new normalization for the left eigenvectors, from which the sensitivities of the left and right eigenvectors can be determined separately and independently for asymmetric viscously damped systems with distinct and repeated eigenvalues. Li et al. [22,23] studied the eigensensitivities of generalized nonviscous damped eigensystems using the algebraic method. Recently, Li et al. [24] developed an algebraic method to calculate the first- and second-order derivatives of eigensolutions of the undamped, damped and nonlinear systems. The algebraic method is a compact and exact method. The algebraic method only requires the mode of interest, and is very efficient to calculate the sensitivity of a few modes. However, the algebraic method needs a matrix decomposition (e.g., $\mathbf{LDL}^T$ decomposition) of the coefficient matrix of the linear system of algebraic equations for each eigensolution sensitivity, which cause the *matrix decomposition problem*, and therefore

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the algebraic method is time-consuming when considering the derivatives of many modes.

Nelson [25] developed an efficient method to obtain the derivatives of mode shapes of undamped systems. The main idea of Nelson's method is to express the derivative of each mode shape as a linear combination of a particular solution and a homogeneous solution of the singularity of the coefficient matrix. The particular solution is suggested to be determined by identifying the element of the corresponding mode shape with the largest absolute value and constraining its derivative to zero. It is well known that each mode shape always satisfies the linear equation of the eigenvector derivatives with singular coefficient matrix and can be therefore considered as a homogeneous solution of its derivative. The arbitrary constraining the derivative of the largest absolute value can be corrected by the calculation of the coefficient of the homogeneous solution. Friswell [26] extended Nelson's method to find the second and higher order derivatives of modes of undamped systems. Later, Friswell and Adhikari [27] extended Nelson's method to viscously damped systems. Guedria et al. [28] studied the second order derivatives of modes of viscously damped systems using Nelson's method. Adhikari and Friswell [29] extended Nelson's method to nonviscously damped systems. Recently, Li et al. [30] developed a unified eigensensitivity method to obtain the sensitivities of eigensolutions of the undamped, viscously, nonviscously damped and nonlinear systems of both distinct and repeated eigenvalues. Although Nelson's method mentioned above gives exact results and only needs the mode shapes of interest, Nelson's method has the matrix decomposition problem similar to the algebraic method since the particular solution needs to be obtained in terms of matrix decomposition and must be resolved for different mode shapes. Therefore, Nelson's method is time-consuming if many modes are considered to eigensensitivity analysis.

Murthy and Haftka [31] surveyed the methods for eigensensitivity analysis of the systems with generalized non-Hermitian matrices. Jankovic [32] gave the analytical solutions for the first and higher order derivatives of eigensolutions of generalized nonlinear eigenproblems. Other methods have been developed for the calculation of the sensitivity of mode shape, including the iterative method [33–35], QR-based method [36,37], Davidson-based method [38], the combination method [39–41], the perturbation method [42,43] and the substructuring method [44–46].

Fox and Kapoor [14] also suggested a modal method, which evaluates the derivative of each mode shape as a superposition of all the mode shapes. This modal method can be used to obtain the eigensensitivities of viscously damped systems using the state-space formulation. Adhikari [47,48] and Adhikari and Friswell [49] developed some original-space modal methods to calculate the sensitivities of eigensolutions of viscously damped systems without using state-space formulations. It should be mentioned that the modal method calculates the eigenvector sensitivities by modal superposition and therefore does not have the matrix decomposition problem. The modal method is widely applied in engineering (see, e.g. in the direct and inverse eigenproblems [8,50–53]). However, in order to guarantee the exact derivative of each mode shape, the modal method needs a superposition of all the mode shapes, which is a significant computational task especially for multiple degree-of-freedom (DOF) engineering problems. Often only the lower order frequencies and associated mode shapes are required for engineering analysis. It implies that approximated sensitivity may be evaluated depending on the number of modal basis vectors.

The corrections to the problem of modal truncation error have been studied by several authors. Wang [54] approximates the contribution of higher (unavailable) modes to the derivatives of mode shapes of undamped systems in terms of a residual-static term solved from the given equation for the mode shape derivative. Liu et al. [55] developed an accurate modal method for undamped systems by combining the mode superposition of lower (available) modes and a convergent series of the influence of higher modes in terms of system matrices and lower modes. Some authors [56–58] have also studied the derivatives of complex mode shapes of viscously damped systems in terms of state-space formulation. These state-space equations based approaches need heavy computational effort since the size of system matrices of state-space equations is double the size of original-space.

The aim of this study is to present an original-space correction modal method to correct the derivatives of complex mode shapes of viscously damped systems. Since the correction modal method only uses original-space eigenproblems (the state-space formulation is avoided), it is efficient. Firstly, based on the Neumann expansion theorem, an explicit expression of modal truncation error can be expressed as a sum of lower modes and system matrices. It will be shown that the modal truncation error of eigensensitivity can be exactly expressed as a convergent series expansion, which can be evaluated by a simple iterative procedure. Then, an original-space correction modal method is presented by expressing the eigenvector sensitivities as the explicit expression of the contribution of the higher modes and the modal superposition of the lower modes. Finally, two case studies will be used to illustrate the engineering application, accuracy and efficiency of the presented method. It will be shown that the proposed method yields good trade-off between the computational accuracy and the computational complexity in terms of the theoretical analysis and numerical studies.

2. Common methods for sensitivity analysis of viscously damped eigensystems

The formulation for modal analysis of viscously damped linear systems can be given by

$$(\lambda_i^2\mathbf{M} + \lambda_i\mathbf{C} + \mathbf{K})\boldsymbol{\varphi}_i = \mathbf{0} \quad \forall i = 1, 2, \ldots, 2N \tag{1}$$

where $\mathbf{M}$, $\mathbf{C}$ and $\mathbf{K} \in \mathbb{R}^{N\times N}$ are, respectively, the mass, damping and stiffness matrices (assume system matrices are symmetric and differentiable with respect to design parameter, p); λ_i is the ith eigenvalue and $\boldsymbol{\varphi}_i$ is the ith complex mode shape (eigenvector). Here we assume that the eigenvalues, which appear in complex conjugate pairs for underdamped systems, are distinct. The damped system cannot be simultaneously decoupled by modal analysis unless it also possesses a full set of classical normal modes (the classical normal modes are undamped mode shapes that are normalized by using the mass normalization). The condition of viscously damped systems to possess classical normal modes, is known as the classically damped system. In 1877, Rayleigh [59] showed that a viscous damping is proportionally damping if the damping matrix is a linear combination of inertia and stiffness matrices. This damping is routinely assumed in engineering applications. Later, Caughey and O'Kelly [60] gave some more restrictive conditions which make viscously damped systems possess normal modes as well. However, there is, of course, no reason why these mathematical conditions must be satisfied. Generally speaking, classical damping means that energy dissipation is almost uniformly distributed throughout the mechanical system. In practical, mechanical systems with two or more parts with significantly different levels of energy dissipation are encountered frequently in engineering designs. To this end, the non-classically damped system is considered in this study, i.e., the concern of this study is when these mathematical conditions are not met (the most general case in engineering applications).

Often the following normalization is adapted to remove the arbitrariness of eigenvectors.