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EXSSA: SSA-based reconstruction of time series via exponential smoothing of covariance eigenvalues

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ABSTRACT

A critical aspect of singular spectrum analysis (SSA) is the reconstruction of the original time series under various assumptions about its underlying structure. This reconstruction depends on the choice of the components from the covariance decomposition of the trajectory matrix. In most applications, this selection is based on the prior knowledge and experience of the researcher and a variety of practical rules. This paper suggests an alternative "fully automated" approach where all components of the covariance decomposition are used via exponential smoothing of the covariance eigenvalues. We illustrate the validity of the proposed approximation via simulations on different data generating processes. A second contribution of the paper is the proposal of a "forecast revision" algorithm which combines SSA with a benchmark. An empirical exercise using four key macroeconomic variables shows how this method can be used to improve the out-of-sample forecasts of any given benchmark model. Our results suggest that the proposed method has the potential to partly automate the use of SSA.

1. Introduction

There now exists a vast body of literature on the topic of singular spectrum analysis (SSA). SSA is a non-parametric technique which incorporates elements of classical time series analysis, multivariate statistics, signal processing, etc. It is favoured by both academics and applied researchers due to its wide applicability: stationary and non-stationary, linear and nonlinear series can all be analysed and predicted using SSA. There are many references across journals from a range of different disciplines on the theory and application of SSA that cannot be reviewed extensively here. As just a few of many possible examples,[1] see the recent book by Hassani and Patterson (2014); the papers by Beneki, Eeckels, and Leon (2011), Hassani, Ghodsi, Silva, and Heravi (2016), Hassani, Heravi, and Zhigljavsky (2009) and Lisi and Medio (1997) on forecasting applications; and the papers by Thomakos, Wang, and Wille (2002) on applications to the realised volatility, Moskvina and Zhigljavsky (2003) on change point detection, Alexandrov, Bianconcini, Dagumb, Maassa, and McElroy (2011) on the problem of trend extraction, Carvalho, Rodrigues, and Rua (2012) on tracking the US business cycle, and Sella and Marchionatti (2012) and Sella, Vivaldo, Groth, and Ghil (2013) on the analysis of economic cycles; see also the paper by Hassani and Thomakos (2010), and the references therein, for a review of the theory and application of SSA to economic and financial time series, including unit roots and cointegration. In addition, Hassani and Mahmoudvand (2013), Hassani, Mahmoudvand, and Zokaei (2011), Hassani, Mahmoudvand, Zokaei, and Ghodsi (2012) and Hassani, Webster, Silva, and Heravi (2015)

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1 The works of the originators of SSA can be found in any of the papers quoted in this section.

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suggest a solution to the window length selection problem. Recently, a series of papers by Khan and Poskitt (2013a,b) has renewed interest on the underlying econometric theory for SSA. Their work has provided a solid theoretical framework for determining when, why and how SSA can perform well. Furthermore, they have provided new results on the selection of the window length and the signal dimension.

SSA can be used to perform various tasks, such as smoothing and trend extraction, among others. However, one of the critical aspects of the method is the reconstruction of the original time series under various assumptions about its underlying structure. This reconstruction depends on the way in which the researcher chooses components from the covariance decomposition of the underlying trajectory matrix. This choice depends on various criteria, and although it can be "automated", it is mostly based on prior knowledge, experience and the subject matter under study. This paper is motivated chiefly by this difficulty, and suggests a new way in which *all* components of the covariance decomposition can be used via the exponential smoothing of the covariance eigenvalues. As with all methodological approaches, our suggested method has both pros and cons. The most obvious arguments in its favour are that it avoids the need to decouple the time series into components (where errors can be made, if only from a lack of experience in working with SSA) and that it uses all components based on the relative "strength" of the eigenvalues. The most obvious argument against it is that it may mix or "confuse" different components that one might want to separate, given that one of the criteria for applying SSA is component separability. However, as separability can only be judged after the analysis and the choice of components is a matter of search and experience, it might not be a bad idea to have a fallback method which can be used as a benchmark for component selection. The proposed approach takes advantage of the structure of the eigenvalues of a symmetric, positive definite matrix, and our analysis confirms that it does have automatic adaptability to various data generating processes. In particular, the researcher now needs to choose only the embedding dimension, i.e., the length of the window used to construct the trajectory matrix, not any of the underlying components. This enables the proposed method to be adapted to either smoothing or modelling for forecasting just by changing the embedding dimension. To the best of our knowledge, this approach is completely new, while being similar in spirit to the work of Álvarez-Meza, Acosta-Medina, and Castellanos-Domínguez (2013) on automatic SSA decomposition.

While we show that our proposed method can indeed be used to match the most significant eigenvalues across a variety of data generating processes, one ought to ask how it is going to be used after decomposition and reconstruction. The main answer that we give tentatively in this paper is that such a method ought to be able to capture most of the significant variation in the underlying series, thus capturing the "core" component of the series. Note that what we call "core" here might be a mixture of separable components, but we are not interested in separability here; instead, we are interested in capturing the part of the series that chiefly explains its direction. Thus, we suggest that our method can be used to enhance standard SSA forecasting applications either by potentially obtaining better sign forecasts, or, as we illustrate later in the paper, by providing auxiliary information to SSA or other benchmark forecasts. We therefore aim to make this method of "automating" SSA a way of including potentially useful information in any forecasting context. In the empirical application in this paper, we use the suggested SSA approach to "revise" the forecasts of a given benchmark model. We show that the SSA with exponential smoothing of eigenvalues can improve the forecasts of a benchmark autoregressive model for the annual growth rate series of: (i) real disposable personal income, (ii) real gross domestic income, (iii) real gross domestic produce, and (iv) the producer price index. As a final note, the results of our application indicate how the proposed method can be used in order to improve the forecasting performance, and in no way indicate which benchmark should be used in other cases. The choice of the forecasting model to be used as a benchmark is left to the discretion of the researcher.

The rest of the paper is organised as follows: Section 2 briefly reviews the SSA reconstruction of a time series and explains the new approach using various examples of DGPs and simulations; Section 3 provides the algorithm and the empirical evidence on the applicability of the new approach to the forecasting of the real disposable personal income, real gross domestic income, real gross domestic product, and producer price index annual growth rates; and Section 4 summarises our conclusions.

2. SSA decomposition and reconstruction

2.1. Exponential smoothing of covariance eigenvalues

Consider the time series $\{X_t\}_{t\in\mathbb{S}}$ that takes values in $\mathcal{R}_X \subseteq \mathbb{R}$. The index set $\mathbb{S}$ can be either $\mathbb{Z}$ or $\mathbb{N}$, thus covering the cases of both stationary and nonstationary time series. Denote the $(n \times L)$ trajectory matrix of the sample $\{X\}_{t=1}^{N}$, with $n \stackrel{\text{def}}{=} N-L+1$, as $\mathbf{T}_X$, and write $\mathbf{T}_X \stackrel{\text{def}}{=} [\mathbf{X}_1, \mathbf{X}_2, \dots, \mathbf{X}_L]$, where each $\mathbf{X}_i$, $i = 1, 2, \dots, L$ is a $(n \times 1)$ column vector. L is the embedding dimension, the length of the window used to construct the trajectory matrix (but note that n is also an embedding dimension, even though traditionally L is used as such).

The $(L \times L)$ sample covariance matrix is then defined as:

$$\mathbf{C}_n \stackrel{\text{def}}{=} n^{-1}\mathbf{T}_X^{\top}\mathbf{T}_X, \tag{1}$$

and the (i, j)th element of $\mathbf{C}_n$, with $i \geq j$, is given by $c_{n,ij} \stackrel{\text{def}}{=} n^{-1}\mathbf{X}_i^{\top}\mathbf{X}_j = n^{-1}\sum_{t=j}^{N-L+j} X_{t+(i-j)}X_t$. In standard notation, denote the spectral decomposition of $\mathbf{C}_n$ by:

$$\mathbf{C}_n \stackrel{\text{def}}{=} \mathbf{V}\boldsymbol{\Lambda}\mathbf{V}^{\top} = \sum_{j=1}^{L} \lambda_j \mathbf{v}_j \mathbf{v}_j^{\top}, \tag{2}$$

with $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_L$. In applications, we would use an estimate of $\mathbf{T}_X$ that is given by:

$$\widehat{\mathbf{T}}_X(r) \stackrel{\text{def}}{=} \mathbf{T}_X\mathbf{Q}(r) = \sum_{j\in\mathcal{I}_r} \widehat{\mathbf{T}}_X(j), \tag{3}$$