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Two methods for the computation of nonlinear modes of vibrating systems at large amplitudes

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Abstract

The aim of this paper is to present two methods for the calculation of the nonlinear normal modes of vibration for undamped nonlinear mechanical systems: the time integration periodic orbit method and the modal representation method. In the periodic orbit method, the nonlinear normal mode is obtained by making the continuation of branches of periodic orbits of the equation of motion. The terms "periodic orbits" means a closed trajectory in the phase space, which is obtained by time integration. In the modal representation method, the nonlinear normal mode is constructed in terms of amplitude, phase, mode shape, and frequency, with the distinctive feature that the last two quantities are amplitude and total phase dependent. The methods are compared on two DOF strongly nonlinear systems.

Keywords: Vibration; Normal mode; Nonlinear structure; Periodic motion

1. Introduction

Extending the concept of normal modes of vibration to the case where the restoring forces contain nonlinear terms, has been a challenge to many authors. This has led to the so-called nonlinear normal modes (NNMs) which have great potential for applications in nonlinear vibrating systems. For instance, a damped system exited by harmonic forcing will have its resonances close to the NNMs, and this is a first obvious motivation for their computation. It is also now established that the knowledge of the NNMs, together with their bifurcations, can be very helpful to understand the dynamics of a nonlinear system [1]. Some important phenomenon such as the localisation of the motion [2], the interaction between modes [3], the pumping of energy of a linear system by a pure nonlinear one [4], can be nicely explained using the NNM concept. The NNMs are also important for the modal controllability of nonlinear systems [5]. Finally, eventhough the principle of superposition does not hold for nonlinear equations, the NNM can be useful to generate effective reduced-order models for multi-degree-of-freedom nonlinear systems [6,7].

Following the pioneer work by Rosenberg [8] on conservative systems, several attempts have been made to develop methods for the calculation of nonlinear normal modes. Without entering into details, we mention here several classes of techniques which have aimed this goal. They are the harmonic balance approach [9–12], the normal form theory [13–15], many perturbation techniques [16,17] such as the famous multiple scale analysis, and the invariant manifold method [18,19] which led to a new definition of NNMs, extending the concept to nonconservative systems. Many of the above mentioned methods are based on some kind of perturbation expansions with a truncature of the series after the first few leading order terms. They have the advantage to provide analytical expressions of the NNMs, but on the other hand, the drawback to be limited to weak nonlinearities or small amplitudes. Is it now evident, see

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[7,20] for instance, that numerical methods should be developed to complete the analytical ones. These numerical methods should be able to explore the NMN at large amplitude of vibration, and possibly, to detect all kind of bifurcations along a NNM. The aim of this paper is to present and compare two approaches that fall in that category. The first one is the time integration of a periodic orbit and the second, the modal representation method. In both of them, we compute a one dimensional family of periodic solutions that are parametrized either by the level of mechanical energy or by the amplitude of the orbits. These families of periodic orbits provide a two dimensional invariant subspace of the phase space which has the property to pass through the equilibrium point (zero amplitude orbits) and to be tangent to the linear mode of the underlying linear system (small amplitude orbits). Accordingly, these families of orbits correspond to the NNM as defined by Shaw and Pierre [18,19].

The paper is organized as follows: the mechanical framework is given in Section 2. Section 3 is devoted to the first method which is the time integration of periodic orbit. The NNMs are computed here by making the continuation of branches of periodic orbits that are parametrized by the energy level. The closed orbits are computed using an "exact energy conserving" time integration algorithm [21]. This transforms the boundary value problem into an algebraic one which depends on a free parameter, and which is solved with a semi-analytical continuation technique: the so-called asymptotic-numerical method (ANM) [22]. This approach provides the period of the oscillations on the NNMs, and a time discrete representation of the orbits. The determination of the stability and of the bifurcation of the NNM is well established with this numerically oriented approach [23]. It will presented in a forthcoming paper. Section 4 is devoted to the second method called "the modal representation method". As in the linear case, an expression is developed for the NNM in terms of the amplitude, mode shape, and frequency, with the distinctive feature that the last two quantities are amplitude and total phase dependent. The dynamics of the periodic response is defined by a one dimensional nonlinear differential equation governing the total phase motion. The period of the oscillations, depending only on the amplitude, is easily deduced. It is established that the frequency and the mode shape provide the solution to a 2π-periodic nonlinear eigenvalue problem from which a numerical Galerkin procedure is developed for approximating the NNMs. This formulation allows us to characterize the similar NNMs. It leads also to an analytical (parametric) expression of the invariant manifold and it permits to compute the NNM even in the case of resonance relations between the eigenvalues of the linearized system. The extension of the formulation in the case of damped autonomous mechanical systems is considered in [24]. Finally, in Section 5, these two methods are compared on a benchmark problem with two Green–Lagrange springs, which is representative of geometrically nonlinear thin structures such as plates and shells.

2. Mechanical framework

In this study, we consider the undamped autonomous nonlinear n degrees of freedom (n-DOF) mechanical system

$$\mathbf{M}\ddot{\mathbf{U}}(t) + \mathbf{F}(\mathbf{U}(t)) = \mathbf{0}, \tag{1}$$

where $\mathbf{M}$ is the mass matrix, and $\mathbf{F}(\mathbf{U})$ is the vector of restoring forces including linear and nonlinear terms. The overdots stand for temporal derivatives. The following assumptions will be made throughout this study:

- H1: $\mathbf{M}$ is a symmetrical positive definite matrix;
- H2: $\mathbf{F}(\mathbf{U}) = \frac{\partial W}{\partial \mathbf{U}}(\mathbf{U})$ where $W(\mathbf{U})$ is a scalar potential energy function of the vector $\mathbf{U} \in R^n$, W has a continuous second derivatives, is positive, and admits a local minimum at $\mathbf{U} = \mathbf{0}$. (As usual $\frac{\partial W}{\partial \mathbf{U}}(\mathbf{U})$ will be denoted $W_{,\mathbf{U}}(\mathbf{U})$).

The conservative system (1) has a first integral corresponding to the conservation of the total energy E, which is the sum of the kinetic energy and the potential energy W i.e.

$$E(\mathbf{U}(t)) = \frac{1}{2}\dot{\mathbf{U}}(t)^{\mathrm{T}}\mathbf{M}\dot{\mathbf{U}}(t) + W(\mathbf{U}(t)). \tag{2}$$

The linear equation

$$\mathbf{M}\ddot{\mathbf{U}}(t) + [W_{,\mathbf{UU}}(0)]\mathbf{U}(t) = 0, \tag{3}$$

where $[W_{,\mathbf{UU}}(0)]$ denotes the Hessian matrix for the function W at $\mathbf{U} = \mathbf{0}$, will be called the underlying linear system (or linearized system) associated with the nonlinear equation (1).

It should be noted that the framework (1) includes the equations of motion of elastic thin structures with geometrical nonlinearities such as shells, plates, beams and cables. The continuous model should be discretised using a classical Ritz or finite element method.

In the following, we focus on the periodic solution of (1). If there is no internal resonance (the eigenfrequencies of the linearized system (3) are no commensurable), the conservative system (1) possesses at least n two dimensional families of periodic solution around the stable origin $\mathbf{U} = \mathbf{0}$. These two dimensional families of periodic orbits allow description of two dimensional invariant manifolds of the phase space, corresponding to the NNM, as defined by Shaw and Pierre [18].

3. Periodic orbits method

The numerical computation of periodic orbits has already been addressed in textbooks [23,25] either for the calculation of isolated orbits or for the continuation of a family of orbits. The most popular method is the so-called shooting method which consists in finding a suitable initial condition, that induces a closed trajectory in the phase space. This leads to a boundary value problem [23] where