Computers and Structures 141 (2014) 46–58

Contents lists available at ScienceDirect

Computers and Structures

journal homepage: www.elsevier.com/locate/compstruc

A three-node triangular element with continuous nodal stress

Yongtao Yang [a], Xuhai Tang [b,*], Hong Zheng [a]

[a] State Key Laboratory of Geomechanics and Geotechnical Engineering, Institute of Rock and Soil Mechanics, Chinese Academy of Sciences, Wuhan, China
[b] Department of Civil and Environmental Engineering, Princeton University, Princeton, USA

ARTICLE INFO

Article history:
Received 19 June 2013
Accepted 1 May 2014
Available online 17 June 2014

Keywords:
Partition of unity method
Hybrid FE-Meshless method
Trig3-CNS
Continuous nodal stress
Mesh distortion
Convergence rate

ABSTRACT

A three-node triangular element with continuous nodal stress (Trig3-CNS) is presented. The present work is the development of the partition-of-unity based 'FE-Meshless' quadrilateral element with continuous nodal stress (Quad4-CNS). This triangular element, Trig3-CNS, is potential to replace Quad4-CNS when the geometry of computational domain is too complex to generate quadrilateral meshes. The three-node triangular element (Trig3) is widely used, however it is always of low accuracy due to its low order of trial functions. By increasing the order of trial functions, Trig3-CNS obtains better accuracy and higher convergence rate as compared to Trig3. The numerical tests in this paper demonstrate that Trig3-CNS has higher tolerance to mesh distortion as compared to the Trig3 and four-node quadrilateral element (Quad4). Another advantage is that the derivative of Trig3-CNS shape function is continuous at nodes, therefore Trig3-CNS is capable of giving smoother solution and calculating nodal stress without any extra smoothing operation.

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1. Introduction

The finite element method (FEM) has been successfully used [1,2] in many fields, however it is well known that the gradient of the FEM with standard DOF is discontinuous on internal element edges, including at the nodes. Extra smoothing operations are frequently required to calculate nodal stress in post processing. Furthermore, the three-node triangular element (Trig3) is constant strain element and is always of low accuracy due to its low order of trial functions [3]. The performance of Trig3 is particularly bad for obtuse angles that appear in distorted meshes. Another widely used finite element, four-node iso-parametric quadrilateral element (Quad4), has also been proved to be extremely sensitive to mesh quality [4]. Due to their sensitivity to the mesh quality, the Trig3 and Quad4 suffer from serious numerical difficulties when the deformation of the material is significantly large [5].

To improve the accuracy of FEM, Liu et al. [6] and other researchers developed the smoothed finite element method (SFEM) [7,8]. In SFEM, the strain is smoothed based on nodes [9] or the edges of elements [10]. Since SFEM is able to provide an upper bound solution for solid mechanics problems, it is complementary to the FEM. Alternatively, Thai-Hoang et al. [11] used a discrete shear gap technique to develop an alpha finite element method, which significantly improves the accuracy of FEM solution.

The present work uses partition of unity method (PUM) [12,13] to construct shape functions. During the last two decades, PUM has been developed and applied in many fields successfully, including solid mechanic [14], fluid mechanic [15] and heat transfer [16,17]. The concept of PUM is also used to develop the hp clouds [18], generalized finite element method (GFEM) [19], particle-partition of unity method [20], numerical manifold method [21,22] and extended finite element method [23]. Furthermore, PUM provides a convenient approach to construct higher order global approximations without adding external nodes, thereby achieving higher accuracy and convergence rate [24].

The present method is an extension of the hybrid FE-Meshless method [5]. The meshless method was first proposed by Belytschko et al. [25], which was named as element free galerkin method (EFG) [26]. The development of meshless method can be found in the work associated to point interpolation method (PIM) [27], reproducing kernel particle method (RKPM) [28], stable particle methods [29] and meshfree local Petrov–Galerkin method (MLPG) [30]. Because meshfree methods do not require construction of a finite element mesh over computational domain, they avoid the problems of mesh updating and mesh distortion, which makes it particularly effective for simulating large deformation problems [31,32]. Furthermore, in the problems of Kirchhoff–Love plate and shell models [33], the continuity conditions between the elements have to be imposed not only on the primary field variable

* Corresponding author.
E-mail address: xuhait@princeton.edu (X. Tang).

http://dx.doi.org/10.1016/j.compstruc.2014.05.001
0045-7949/Published by Elsevier Ltd.

but also on its derivative. Meshless methods are able to generate smooth shape functions, therefore they have already been successfully used in the simulation of Kirchhoff–Love plate and shell models [33]. However, it is well known that meshless method requires extra operations to treat the essential boundary conditions due to the absence of the Kronecker delta property of shape functions. To overcome the drawbacks of meshfree methods, some partition of unity based (PU-based) hybrid FE-Meshless methods have been proposed, such as the hybrid FE-Meshless four-node quadrilateral element (FE-LSPIM QUAD4) [5,34]. These PU-based hybrid FE-Meshless methods inherit better accuracy, higher convergence rate, and high tolerance to mesh distortion from the meshless methods [35], while preserving the Kronecker-delta property of the standard iso-parametric displacement models.

In the present work, a three-node triangular element with continuous nodal stress (Trig3-CNS) is introduced. Trig3-CNS can be regarded as the development of the four-node hybrid FE-Meshless method with continuous nodal stress (Quad4-CNS) [35]. This triangular element, Trig3-CNS, is potential to replace Quad4-CNS when the geometry of computational domain is too complex to generate quadrilateral meshes. The triangular element is frequently used to simulate multiple-crack propagation, where the mesh is updated to match the propagating crack surfaces at each computational step [36,37]. Trig3-CNS also can be regarded as the development of the PU-based three-node triangular element with discontinuous nodal stress (Trig3-DNS). Compared to Trig3-DNS, Trig3-CNS is capable of giving smoother solution and calculating nodal stress without any extra smoothing operation.

2. Partition of unity method

Consider a bounded domain, V, in two dimensions. The PUM is constructed using a set of non-negative weight functions (support functions), $\{w_1(\boldsymbol{x}), w_2(\boldsymbol{x}), \ldots, w_n(\boldsymbol{x})\}$, which sum to 1,

$$\sum_{i=1}^{n} w_i(\boldsymbol{x}) \equiv 1, \tag{1}$$

where $\boldsymbol{x}$ is a domain point with a position coordinate (x,y) and n is the total number of nodes in domain V.

It is noticed that, in the early version of PUM [12], the requirement for $w_i(\boldsymbol{x}) \geqslant 0$ is discarded, which is slightly different from the standard statements of the partition of unity theorem [38]. The approximation on the domain V is defined as:

$$u^h(\boldsymbol{x}) = \sum_{i=1}^{n} w_i(\boldsymbol{x}) u_i(\boldsymbol{x}), \tag{2}$$

where $u_i(\boldsymbol{x})$ is the nodal approximation associated with the node i and $u^h(\boldsymbol{x})$ is the global approximation. The choice of weight functions and nodal functions are flexible in PUM. In hybrid FE-Meshless Quad4, the formulas for constructing the shape functions of Quad4 are chosen as weight functions [5]. The nodal approximation can be constructed using many approaches, such as least-squares approximation [39] and radius function based approximation [40]. In the present work, to ensure the Kronecker-delta property of the shape functions, the constrained and orthogonalizing least-squares method (CO-LS) is applied to construct nodal approximation [35].

3. Formulation of three-node triangular element with continuous nodal stress (Trig3-CNS)

First the node patch of the node i is defined by Ω_i, as shown in Fig. 1. Here, node i is called central node, and other nodes in domain Ω_i are called satellite nodes. The element support domain

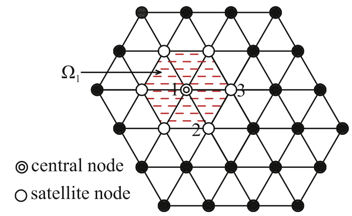

Fig. 1. Node patch of node 1.

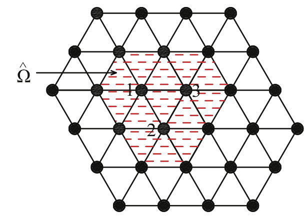

Fig. 2. Element support domain.

$\widehat{\Omega}$ is the union of the three node patches $\widehat{\Omega} = \cup_{i=1}^{3} \Omega_i$ as shown in Fig. 2.

Now we consider a triangular domain Ω described by three nodes $\{P_1\ P_2\ P_3\}$ and introduce an arbitrary point $P(\boldsymbol{x})$ with the coordinates $\boldsymbol{x} = (x,y)$. According to the concept of PUM [12], in the triangular domain Ω, the Trig3-CNS global approximation $u^h(\boldsymbol{x})$ can be represented in the following form:

$$u^h(\boldsymbol{x}) = w_1(\boldsymbol{x})u_1(\boldsymbol{x}) + w_2(\boldsymbol{x})u_2(\boldsymbol{x}) + w_3(\boldsymbol{x})u_3(\boldsymbol{x}), \tag{3}$$

where $w_i(\boldsymbol{x})$ and $u_i(\boldsymbol{x})$ are the weight functions and the nodal approximations associated with node i.

3.1. Trig3-CNS: Construction of weight function

The area coordinates is used to construct the weight functions of Trig3-CNS. The transformation of the area coordinate is defined as:

$$\begin{aligned} \begin{bmatrix} 1 \\ x \\ y \end{bmatrix} &= \begin{bmatrix} 1 & 1 & 1 \\ x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \end{bmatrix} \begin{bmatrix} L_1 \\ L_2 \\ L_3 \end{bmatrix}, \quad \begin{bmatrix} L_1 \\ L_2 \\ L_3 \end{bmatrix} \\ &= \frac{1}{2A} \begin{bmatrix} x_2y_3 - x_3y_2 & y_2 - y_3 & x_3 - x_2 \\ x_3y_1 - x_1y_3 & y_3 - y_1 & x_1 - x_3 \\ x_1y_2 - x_2y_1 & y_1 - y_2 & x_2 - x_1 \end{bmatrix} \\ &\underline{\underline{\mathrm{def}}} \frac{1}{2A} \begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix} \begin{bmatrix} 1 \\ x \\ y \end{bmatrix}, \end{aligned} \tag{4}$$

in which

$$2A = \det \begin{bmatrix} 1 & 1 & 1 \\ x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \end{bmatrix}, \quad L_1 + L_2 + L_3 = 1. \tag{5}$$

Similar to constructing the shape function of triangular elements in the thin plate theory [1], the weight functions of Trig3-CNS are written as

$$w_1(\boldsymbol{x}) = L_1 + L_1^2L_2 + L_1^2L_3 - L_1L_2^2 - L_1L_3^2, \tag{6}$$

$$w_2(\boldsymbol{x}) = L_2 + L_2^2L_3 + L_2^2L_1 - L_2L_3^2 - L_2L_1^2, \tag{7}$$

$$w_3(\boldsymbol{x}) = L_3 + L_3^2L_1 + L_3^2L_2 - L_3L_1^2 - L_3L_2^2. \tag{8}$$