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# A unified formulation for finite element analysis of piezoelectric adaptive plates

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**Abstract**

This paper presents some finite elements for the dynamic analysis of laminated plates embedding piezoelectric layers based on the principle of virtual displacements (PVD) and an unified formulation. The description of the unknowns allows to keep the order of the expansion of the variables along the thickness direction as a parameter of the model and at the same time to perform both *equivalent single layer* (ESL) and *layer-wise* (LW) descriptions of the state mechanical variables. The full coupling between the electric and mechanical fields is considered; thus, the electric potential is taken as a state variable of the problem and is assumed LW with an order of the expansion that goes from 1 to 4 as the displacement field. In case of a ESL model the ZZ-function of Murakami has been introduced to recover the *zig-zag* form of the displacement field. Combining all the possible parameters, up to 12 different finite elements are addressed. Numerical results have been given for the free-vibrations frequencies of simply supported plates embedding piezoelectric layers. The lamination of the pure elastic layers has been limited to cross-ply in order to compare numerical results to analytical ones obtained by a Navier-type solution based on the same formulation.


*Keywords:* Piezoelectric plates; Multilayered plates; Finite elements; Unified formulation; Higher order theories; Free-vibration frequencies

## 1. Introduction

Due to their potential applications, during the last decades, significant efforts have been devoted to the research on the so-called *smart* structures. Such structures differ from the conventional ones by the presence of elements able to perform as actuators and/or sensors, allowing the structure itself to adapt and/or sense to the external environment. This capability leads to a wide range of applications, in particular in the aerospace field such as vibration suppression, shape adaption of aerodynamic surfaces, noise reduction, precision positioning of antennas, aeroelastic control of lifting surfaces and shape control of optical devices. Even if a variety of different materials can be utilized in smart structures, only piezoelectric ones have shown the capability to perform effectively both as actuators and sensors. Another advantage of piezoelectric materials which explain their widespread use in structural engineering applications is the simple integration with composite structures. Composite structures integrating piezoelectric materials offer the possibility to combine the low density, superior mechanical and thermal properties of composite materials along with sensing, actuation and control. In so doing, many aerospace applications of piezomaterials are concerned with composite structures which are intrinsically multilayered made. Due to the discontinuity of the mechanical and electrical properties at the layer interfaces, multilayered structures embedding piezoelectric layers require accurate electromechanical modeling. In order to anticipate the behavior of smart structures embedding piezoelectric materials and prevent failure mechanisms, an accurate description of the mechanical and electrical fields in the layers is essential, see [1–3].

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Multilayered structures exhibit higher transverse shear and transverse normal flexibilities with respect to in-plane deformability along with a discontinuity of the mechanical and electrical properties in the thickness direction. These peculiarities require the displacement field and the transverse stresses to satisfy some conditions which have been summarized by Carrera [4,5] with the acronym $C_z^0$. In particular the displacement field $\boldsymbol{u}$ should be able to describe sudden changes of slope in correspondence of layer interfaces. This is known as the *zig-zag* effect (ZZ). Although in-plane stresses $\boldsymbol{\sigma}_p$ can be discontinuous, the *Cauchy* theorem demands the continuity of the transverse stresses $\boldsymbol{\sigma}_n$ and the electric transverse displacement $D_z$ and potential $\Phi$ through the thickness. This is indicated as *interlaminar continuity* (IC).

For these reasons, the use of plate theories based on the extension of the Kirchhoff hypotheses (classical lamination theory, CLT) and Reissner–Mindlin (first order shear deformation theory, FSDT) for the static and dynamic analysis of piezoelectric plates can lead to inaccurate results. Assessments and reviews on the modeling and analysis of multilayered piezoelectric plates can be found in papers [6,7].

The interest of this paper has been turned towards the calculation of free-vibrations frequencies of multilayered plates embedding piezoelectric layers.

Three-dimensional coupled [8] solution for the free-vibrations analysis of simply supported piezoelectric plates has been proposed in the literature. Several two-dimensional analytical solutions have been also presented, among the others the one by Mitchell and Reddy [9] in which a third-order shear deformation theory (TSDT) has been associated with a layer-wise quadratic electric potential approximation can be remembered. Even if three-dimensional and two-dimensional analytical models have reached an impressive accuracy, the solution of practical problems often demand the use of computational methods such as the finite element method. In literature solid, shell, plate and beam finite elements for the analysis of piezoelectric plates can be found. Interested readers are addressed to the survey on finite element modeling of adaptive structures by Benjeddou [1].

The present paper presents some finite elements for the analysis of multilayered composite plates embedding piezoelectric layers based on the principle of virtual displacements (PVD) and the unified formulation introduced by Carrera [4]. This formulation has already shown all its potentiality as a base for finite elements in the mechanical [10,11] and thermoelastic analysis [12] of multilayered plates. One of the most interesting features of the unified formulation consists in the possibility to keep the order of the expansion of the state variables along the thickness of the plate as a parameter of the model. In so doing, both *equivalent single layer* (ESL) and *layer-wise* (LW) descriptions of the variables are allowed. The displacement unknowns are expanded up to the fourth order through a set of functions $F_\tau$ that depend only on the thickness coordinate $z$. These functions can be simple Taylor polynomials or a manipulation of the Legendre ones respectively for an ESL or a LW description of the laminate. In case of a ESL model the *zig-zag* form has been recovered introducing the ZZ-function of Murakami [13]. The electrical potential assumption has been limited to a LW description and the order of the expansion goes from 1 to 4 as the displacement field. This feature is particularly suitable since electric degrees of freedom (dofs) are often avoided for plates elements [14] or a simple through-thickness linear variation [15,16,17,18] is assumed for the electric potential. One of the few plate elements that consider a quadratic assumption for the electric potential is the work by Carrera [19]. The linear through-thickness hypothesis for the electric potential, as demonstrated by Benjeddou [1], neglects systematically the contribution of the induced potential leading to a partial electromechanical coupling. With the present formulation, the errors induced by the linear assumption of the electric potential can be easily pointed out. In fact, combining all the possible parameters, up to 12 theories are addressed in this work.

Since all the finite elements proposed in this work are based on the PVD, the $C_z^0$ requirements are not completely satisfied. In particular, the IC condition on the transverse stresses is fulfilled with a post-processing procedure and that of the transverse electric displacement is not considered.

Finite elements based on Reissner mixed variational principle for the analysis of piezolaminated plate structures have been proposed in [20]. In that work, transverse stresses as well as transverse electrical displacement have been taken as additional primary variables fulfilling the interlaminar conditions.

## 2. Geometry and material assumptions

A multilayered plate is a laminate obtained by stacking rectangular layers until the desired thickness and stiffness are reached. Generally each layer can be made of any kind of material (piezoelectric or purely elastic). Laminae are considered homogeneous, perfectly bonded with each other and operating in the linear elastic range.

To analyze such a typology of structures, a Cartesian coordinate system $x, y, z$ referred to the middle surface is used and it can be seen in Fig. 1 for a plate whose dimensions and thickness are $a$, $b$ and $h$. This system is called laminate coordinate system. For the analysis of composite materials, a second coordinate system is needed: the material coordinate system. This is given for each lamina and is commonly indicated by the axes 1, 2, 3. The material coordinate 1-axis is taken to be parallel to the fiber direction, the 2-axis is transverse to the fiber direction in the plane of the lamina while the 3-axis is perpendicular to the lamina. All the material data such as Young moduli and piezoelectric coefficients, are given in the material reference system instead, the analysis of the structure is made in the laminate coordinate system. This implies that the