Computers and Structures 140 (2014) 14–22

Contents lists available at ScienceDirect

Computers and Structures

journal homepage: www.elsevier.com/locate/compstruc

Spurious modes in geometrically nonlinear small displacement finite elements with incompatible modes

Theodore Sussman [a], Klaus-Jürgen Bathe [b,*]

[a] ADINA R & D, Inc., Watertown, MA 02472, United States
[b] Massachusetts Institute of Technology, Cambridge, MA 02139, United States

ARTICLE INFO

Article history:
Received 15 March 2014
Accepted 16 April 2014
Available online 22 May 2014

Keywords:
Incompatible finite elements
Spurious modes
Small displacements
Small strains
Geometrically nonlinear analysis
Critical strain value

ABSTRACT

We demonstrate the existence of spurious modes in finite elements with incompatible modes when a geometrically nonlinear displacement analysis with small displacements and strains is performed. The spurious modes are a direct consequence of the incompatibility of the elements with displacement boundary conditions. We derive a critical compressive strain condition analytically, and show that the critical strain can be quite small, with small displacements, if the geometric aspect ratio of the elements is large but still practical. In numerical examples we give further insight and results in correspondence with the analytical theory, and demonstrate that spurious modes can be triggered in practical small strain analyses when using elements with incompatible modes.

1. Introduction

The two-dimensional 4-node and three-dimensional 8-node incompatible modes finite elements (here now referred to as ICM elements) have been proposed in Ref. [1], improved for element geometric distortions in Ref. [2] and extensively used for many years. As the formulation of the ICM elements is quite well known, we do not give the formulation details here; for a general description of the elements, see for example Ref. [3]. The key advantage of these elements, as compared with the elements not containing incompatible modes, is that the ICM elements do not exhibit shear locking when subjected to pure bending. The undistorted ICM elements pass the linear analysis patch tests, and, with proper corrections, the distorted elements also pass these tests [2,3].

The elements are in fact a special case of the 'enhanced strain elements' (referred to as EAS elements) proposed in Ref. [4] and extensively studied, see for example Refs. [5–11] and the references therein. The concept of enhancing the strain assumptions in finite elements is quite appealing to avoid locking phenomena but it was also shown that the EAS elements, and hence ICM elements, are in fact unstable in *large strain* conditions [8–10]. To remedy the behavior of the elements, a stabilization technique was proposed [11] with the use of deformation dependent stabilization parameters. While perhaps useful, however, in our developments of finite element procedures we prefer to not use such factors [3,12].

Although the ICM elements are perhaps best suited to model pure bending, in practice the elements are used in very complex nonlinear analyses in which the elements are subjected to a wide variety of stresses and boundary conditions. Thus it is valuable to understand the behavior of these elements in as many situations as possible. In fact, the present paper is motivated by an attempt to explain the unexpected and unphysical results obtained in the analysis of a large industrial problem involving only small strains.

The purpose of the present paper is to demonstrate the existence of spurious modes (displacement patterns with zero or negative energy) in meshes of geometrically nonlinear ICM elements subjected to *only small displacement and small strain conditions*. The spurious modes are associated with the incompatibility of the incompatible modes with the boundary conditions, and the spurious modes are triggered when the strains reach a certain critical value. The critical value is highly dependent on the element geometric aspect ratios, with elements having a large aspect ratio giving a small critical value.

In Section 2, we derive the tangent bending modulus of a two-dimensional undistorted rectangular 4-node ICM element, assuming geometric nonlinear conditions, and a linear elastic material with zero Poisson's ratio. The theoretical results show that even in small displacements only, the element can become unstable.

In Section 3, we present some numerical examples that illustrate and give insight into the derived theoretical results, and that

* Corresponding author. Tel.: +1 6179265199.
E-mail address: kjb@mit.edu (K.J. Bathe).

http://dx.doi.org/10.1016/j.compstruc.2014.04.004
0045-7949/

show the spurious modes for various element assemblages in two- and three-dimensional analyses. We also include the results of a practical analysis case.

Finally we give some closing remarks in Section 4.

2. Derivation of the element critical strain criterion

In the derivation of the critical strain condition, we make various simplifying assumptions in order to have a clear derivation and obtain insight into the reasons for the instability to occur. These assumptions will be introduced during the derivation.

The principle of virtual work for the total Lagrangian formulation is [3]

$$\delta W = \int S_{ij}\delta\varepsilon_{ij}\, dV \tag{1}$$

where δW is the virtual work, S_{ij} is the 2nd Piola–Kirchhoff stress tensor, referred to the original configuration, ε_{ij} is the Green–Lagrange strain tensor, referred to the original configuration and the integral is performed over the original volume. For brevity, we have dropped the left super- and subscripts commonly used to indicate the time and reference configuration [3].

The increment in the principle of virtual work can be written

$$\Delta\delta W = \int \left(\frac{\partial S_{ij}}{\partial \varepsilon_{kl}}\Delta\varepsilon_{kl}\delta\varepsilon_{ij} + S_{ij}\Delta\delta\varepsilon_{ij}\right) dV \tag{2}$$

where the Δ denotes an increment.

In the following we assume small strain conditions with linear isotropic elasticity, Poisson's ratio equal to zero and planar deformations in the x_1–x_2 plane. Eq. (2) becomes

$$\begin{aligned}\Delta\delta W = \int (&E(\Delta\varepsilon_{11}\delta\varepsilon_{11} + \Delta\varepsilon_{22}\delta\varepsilon_{22}) + 4G\Delta\varepsilon_{12}\delta\varepsilon_{12} + S_{11}\Delta\delta\varepsilon_{11} \\ &+ S_{22}\Delta\delta\varepsilon_{22} + 2S_{12}\Delta\delta\varepsilon_{12})\, dV\end{aligned} \tag{3}$$

in which $S_{11} = E\varepsilon_{11}$, $S_{22} = E\varepsilon_{22}$, $S_{12} = 2G\varepsilon_{12}$ and the symmetries of S_{ij} and ε_{ij} are employed.

Using the definition $\varepsilon_{ij} = \frac{1}{2}(u_{i,j} + u_{j,i} + u_{k,i}u_{k,j})$ where $u_{i,j} = \frac{\partial u_i}{\partial x_j}$, u_i are the displacement components, x_i are the (original) coordinate components, and k is an index used in conjunction with the Einstein summation convention, we obtain for small strains

$$\varepsilon_{11} \approx u_{1,1}, \varepsilon_{22} \approx u_{2,2}, \quad \varepsilon_{12} \approx \frac{1}{2}(u_{1,2} + u_{2,1}) \tag{4a, b, c}$$

$$\Delta\varepsilon_{11} \approx \Delta u_{1,1}, \quad \Delta\varepsilon_{22} \approx \Delta u_{2,2}, \quad \Delta\varepsilon_{12} \approx \frac{1}{2}(\Delta u_{1,2} + \Delta u_{2,1}) \tag{5a, b, c}$$

$$\begin{aligned}&\Delta\delta\varepsilon_{11} = \Delta u_{1,1}\delta u_{1,1} + \Delta u_{2,1}\delta u_{2,1}, \quad \Delta\delta\varepsilon_{22} = \Delta u_{1,2}\delta u_{1,2} + \Delta u_{2,2}\delta u_{2,2}, \\ &\Delta\delta\varepsilon_{12} = \tfrac{1}{2}(\Delta u_{1,1}\delta u_{1,2} + \Delta u_{2,1}\delta u_{2,2} + \Delta u_{1,2}\delta u_{1,1} + \Delta u_{2,2}\delta u_{2,1})\end{aligned} \tag{6a, b, c}$$

so that Eq. (3) becomes

$$\begin{aligned}\Delta\delta W = &\int E(\Delta u_{1,1}\delta u_{1,1} + \Delta u_{2,2}\delta u_{2,2})\, dV + \int G(\Delta u_{1,2}\delta u_{1,2} \\ &+ \Delta u_{1,2}\delta u_{2,1} + \Delta u_{2,1}\delta u_{1,2} + \Delta u_{2,1}\delta u_{2,1})\, dV \\ &+ \int S_{11}(\Delta u_{1,1}\delta u_{1,1} + \Delta u_{2,1}\delta u_{2,1})\, dV + \int S_{22}(\Delta u_{1,2}\delta u_{1,2} \\ &+ \Delta u_{2,2}\delta u_{2,2})\, dV + \int S_{12}(\Delta u_{1,1}\delta u_{1,2} + \Delta u_{2,1}\delta u_{2,2} \\ &+ \Delta u_{1,2}\delta u_{1,1} + \Delta u_{2,2}\delta u_{2,1})\, dV\end{aligned} \tag{7}$$

We now consider the single rectangular two-dimensional 4-node ICM element shown in Fig. 1. This element is first subjected to a homogeneous deformation that produces the stresses S_{11},

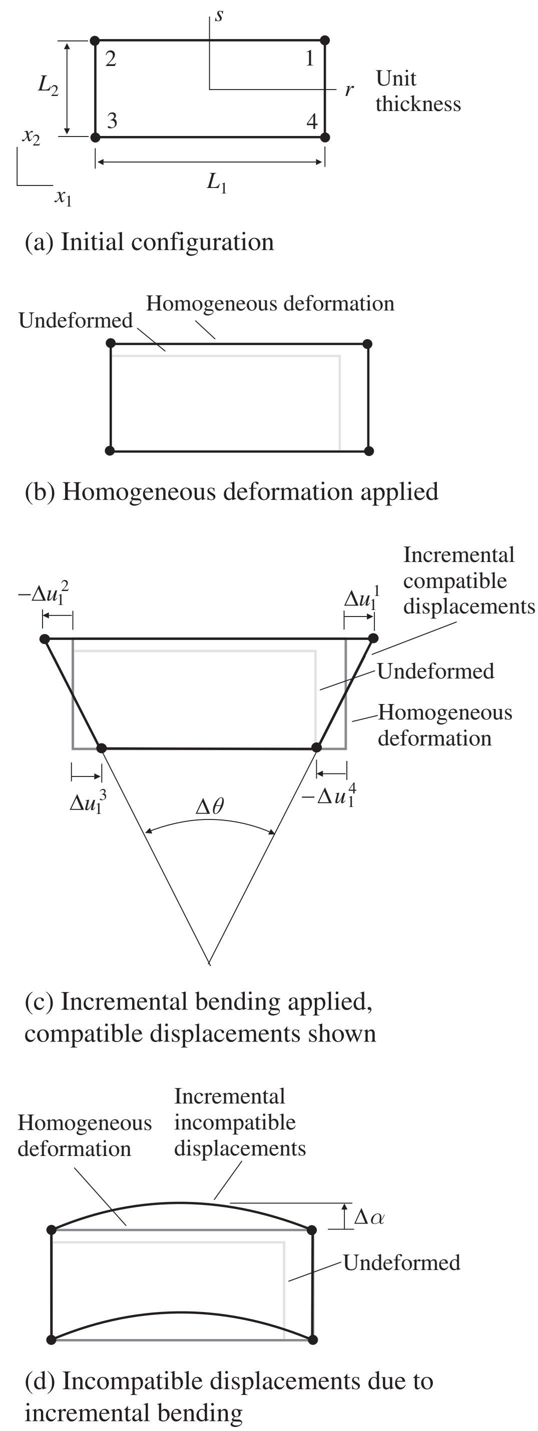

Fig. 1. 2D 4-node ICM element under homogeneous deformation and incremental bending.

S_{22}, S_{12}. Since this deformation is homogeneous, the incompatible modes in the element are not triggered.

Now consider the following incremental nodal displacements: $\Delta u_1^1 = \Delta u_1^3 = \frac{L_2}{4}\Delta\theta$, $\Delta u_1^2 = \Delta u_1^4 = -\frac{L_2}{4}\Delta\theta$ in which the superscript denotes the node number and in which $\Delta\theta$ controls the magnitude of the displacement. Here $\Delta\theta$ represents an incremental rotation as shown in the figure (assuming small strains, the original value of L_2 is used instead of the deformed value). The incremental nodal displacements cause an internal incremental displacement within the element of