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Constraint paths in non-linear structural optimization

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ABSTRACT

Optimization of significantly non-linear structures is a demanding task. The paper discusses how boundaries of the feasible region can be followed as generalized equilibrium paths in parametric space, reflecting engineering demands on stiffness, strains and stability. Solutions on the constraint paths are then evaluated with respect to any chosen objective function. For few design parameters, this approach is efficient and robust. This is demonstrated for a pre-stressed pressurized membrane of three parameters, showing several constraint paths for the problem, and indicating how these are used in optimization. The view is often closer to engineering design analyses than the mathematical optimization settings, which often has problems in handling stability constraints.

1. Introduction

Structural optimization has developed rapidly. The methods seek a structure which fulfills specified functional requirements with a minimum amount of material, or gives the best possible function for a specific amount [1–4, and many others].

The formulation of a mathematical optimization problem defines an objective function to be minimized. Equality and inequality constraints limit some response quantities, but also introduce bounds for the design variables. Delimited by these constraints, a space is investigated for the best possible solution. The basic setting of a mathematical optimization problem is thereby commonly seen as the search for:

$$\min_{\xi} f(\xi) \tag{1}$$

where a chosen objective function is minimized under the constraints:

$$\boldsymbol{h}_1(\xi) = \boldsymbol{0} \tag{2}$$

$$\boldsymbol{h}_2(\xi) \leqslant \boldsymbol{0} \tag{3}$$

and ξ is a set of design variables, the number of which is strongly dependent on the setting. Normally, the optimal solution to the problem will activate at least one inequality constraint in $\boldsymbol{h}_2(\xi)$, i.e., be exactly on the border of violating it.

Structural design will normally define functional requirements on the solution, in the form of strain, stiffness and stability criteria. The criteria are typically based on results from simulations for a discretized structural model. The discrete values are then either included in the design variables ξ or implicitly dependent on them through, e.g., a set of established equilibrium equations. The number of design variables is rather low when a set of global parameters are used to define the design, and high when one design parameter is related to each of the elements or nodes in the structural model. The number of design variables in relation to the number of variables in the discretized structural model has effects on the choice of an appropriate optimization algorithm [3].

For structures where non-linear responses or instabilities are important aspects, the optimization problem is complex [5–8]. This is partly due to the more involved expressions in the structural model description. Another complication is related to the non-uniqueness in response to loading, even if this can occasionally be the objective, as in [9]. The optimization of non-linear structures has been primarily aimed at trusses and shells, and at physically non-linear structures [10]. The non-linearities can in most published cases be considered as rather mild. Very few publications describe optimization settings where the non-linearity leads to marked changes in qualitative behavior of the studied structure. Such significantly non-linear structures will cause problems for the optimization algorithms, due to their non-unique response to loading. Practically, they are also demanding, as they normally need step-wise increments to reach the solution for a specified loading.

The present work is directed towards the optimization of significantly non-linear and instability affected structures, with a low number of design variables. As an illustration of the method, the optimization of a pneumatically pressurized flat membranes has been chosen. For this problem, a few geometric measures, a material parameter and a pre-stretch variable are relevant design

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parameters. Previous work [11,12], has shown that these parameters significantly affect the whole pressurization response of the membrane. The modelling assumptions and methods are given in [11,13], but are briefly described below.

The optimization problem is here focussing primarily on the constraints. These are mapped onto the design space, reflecting the fact that the functional requirements are a necessary objective for the designer, whereas strict optimality is often less fundamental. In relation to the setting above, the main calculations can thereby be seen as a fulfilment of the equality constraints, and successive activations of a subset of the inequality constraints, i.e., symbolically as:

$$\boldsymbol{h}_1(\xi) = \boldsymbol{0} \tag{4}$$

$$\boldsymbol{h}_{2,i}(\xi) = \boldsymbol{0} \tag{5}$$

the solution of which gives a constraint path which constitutes a potential part of the boundary of the feasible design domain. Activating all inequality constraints systematically will thereby delimit the feasible region. The feasible side of the constraint path is normally obvious.

When the feasible region is outlined by the combination of the constraint paths, the optimal solution can be chosen. As such solutions will normally be situated on the boundary of the feasible region, all solutions on the relevant constraint paths are easily evaluated with respect to any chosen objective function, Eq. (1). It is obvious that several objective functions can be simultaneously evaluated for all such solutions, giving the basis for the choice of final design.

In a structural design context, fulfilling the stated constraints leads to an augmented equilibrium problem. Such problems can be solved by generalized path-following, used for evaluation of the parametric sensitivity in instability behavior in [14,15]. By following these paths in the parametric space, the incremental solutions of non-linear equilibrium for each parametric instance can be avoided. A similar aim is recently formulated as a need to follow parameterized instability states [16,17]. The view has some similarity to the response surface methodology [18], and to proper general decompositions [19].

2. Mechanical formulation

2.1. Basic form

The geometrical description of the pressurized membrane assumes a continuous, thin surface structure. The material is seen as incompressible, isotropic, and hyper-elastic. Strains and stresses are described by the right Cauchy–Green deformation tensor $\boldsymbol{C}$, and the second Piola–Kirchhoff stress tensor $\boldsymbol{S}$, both quantities constant over the thickness. Introducing conditions of incompressibility, and a local plane-stress situation, a hyper-elastic model is given by [20]:

$$\boldsymbol{S} = -\rho \boldsymbol{C}^{-1} + 2\frac{\partial W}{\partial \boldsymbol{C}} \tag{6}$$

where W is a strain energy density function, and ρ is seen as a Lagrange multiplier. With the incompressibility assumption expressed in the third strain invariant $I_3(\boldsymbol{C}) = 1$, a Mooney–Rivlin form defines the stress–strain relation from two constitutive constants [20]:

$$W = c_1(I_1(\boldsymbol{C}) - 3) + c_2(I_2(\boldsymbol{C}) - 3) \tag{7}$$

with the first and second invariants of the tensor $\boldsymbol{C}$. Different relations between the two constitutive constants are here represented as a ratio $k = c_2/c_1$, with the demand that $c_1 + c_2 = G/2$, where G is the linear shear modulus [20].

2.2. Element expressions

The element used in the discretization of the large deflection space membrane simulations is a flat, linearly interpolated constant strain triangle [11]. Element nodal translations are extracted from an N_u-dimensional vector of global structural displacements $\boldsymbol{u}$. Strain components are evaluated as non-linearly dependent on nodal displacements, but constant over the element volume. The stress tensor components lead to expressions for the vector of structural internal forces $\boldsymbol{f} = \boldsymbol{f}(\boldsymbol{u})$. Similar expressions also give the element contributions to the tangent stiffness matrix.

An outward normal over-pressure ψ is introduced by a compressible gas with zero density, giving the structural external force vector $\boldsymbol{p} = \boldsymbol{p}(\boldsymbol{u}, \psi)$, where displacements $\boldsymbol{u}$ affect area and orientation of the elements.

2.3. Equilibrium equations

A quasi-static one-parametric pressure loading problem seeks solutions $(\boldsymbol{u}, \psi)$ satisfying the structural residual equilibrium equations:

$$\boldsymbol{F}(\boldsymbol{u}, \psi) \equiv \boldsymbol{f}(\boldsymbol{u}) - \boldsymbol{p}(\boldsymbol{u}, \psi) = \boldsymbol{0} \tag{8}$$

where $\boldsymbol{F}$, $\boldsymbol{f}$, $\boldsymbol{p}$, and $\boldsymbol{u}$ are of dimension N_u, and ψ is the scalar over-pressure. This system gives solutions in the form of one-dimensional pressure-deflection curve segments, with intersections possible only at critical states.

The differential relation corresponding to Eq. (8) is:

$$\delta \boldsymbol{F} = \frac{\partial \boldsymbol{f}}{\partial \boldsymbol{u}}\delta \boldsymbol{u} - \frac{\partial \boldsymbol{p}}{\partial \boldsymbol{u}}\delta \boldsymbol{u} - \frac{\partial \boldsymbol{p}}{\partial \psi}\delta \psi = \left(\boldsymbol{K} - \boldsymbol{K}_p\right)\delta \boldsymbol{u} - \delta \psi \frac{\partial \boldsymbol{p}}{\partial \psi} \tag{9}$$

which gives a tangent stiffness matrix containing a load-dependent term, cf. [11,21]. The structural tangent stiffness matrix is symmetric if the membrane is suitably closed.

The formulation used for the equilibrium problem is restricted to problems where the internal forces are uniquely defined from the current displacements and parameters. Uniqueness in response to a specific loading is not required, however, as the algorithm is able to identify and connect solutions of equal degrees of instability. Non-unique responses, however, demand extra care in the interpretation of results, cf. the examples below.

3. The multi-parametric setting

3.1. Basic form

In the present context of seeking constraint paths for the variables ξ, additional parameters are introduced to allow the investigation of the parameter dependence in response [15]. With N_λ parameters in $\boldsymbol{\lambda} = [\psi, \xi^{\mathrm{T}}]^{\mathrm{T}}$, an augmented system is established as:

$$\boldsymbol{G}(\boldsymbol{z}) \equiv \boldsymbol{G}(\boldsymbol{u}, \boldsymbol{\lambda}) \equiv \begin{pmatrix} \boldsymbol{F}(\boldsymbol{u}, \boldsymbol{\lambda}) \\ \boldsymbol{g}(\boldsymbol{u}, \boldsymbol{\lambda}) \end{pmatrix} = \boldsymbol{0} \tag{10}$$

where the top part corresponds to the equilibrium equations in Eq. (8), and the N_g augmenting equations $\boldsymbol{g}(\boldsymbol{u}, \boldsymbol{\lambda})$ define the subset of equilibrium solutions fulfilling Eqs. (4) and (5). The solutions to the augmented system consists of manifolds of dimension $N_\lambda - N_g$, and contains displacements as well as design variables. The present work aims at problems where $N_g = N_\lambda - 1$ giving solution curve segments, or $N_g = N_\lambda - 2$ giving solution surface patches. This implies that only problems with rather few design parameters are treatable with the present viewpoint.

As the solutions to Eq. (10) are continuous in the parametric space, they can give a sequence of responses to a specified loading for a variable structural design without tracking the whole