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# Dynamic stiffness formulation for composite Mindlin plates for exact modal analysis of structures. Part II: Results and applications

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ABSTRACT

The dynamic stiffness method for composite plate elements based on the first order shear deformation theory is implemented in a program called DySAP to compute exact natural frequencies and mode shapes of composite structures. After extensive validation of results using published literature, DySAP is subsequently used to carry out exact free vibration analysis of composite stringer panels. For each example, a finite element solution using NASTRAN is provided and commented on. It is concluded that the dynamic stiffness method is more accurate and computational efficient in free vibration analysis than the traditionally used finite element method.



## 1. Introduction

The investigation of free vibration behaviour of thin-walled composite structures plays an important role in structural design. Amongst many other applications, the natural frequencies and mode shapes are essentially required to avoid resonance, to predict the dynamic response and to study sound transmission. For thin composite structures, bending or out of plane vibration occurs at relatively lower frequencies than the inplane or membrane ones. For this reason, bending vibration has been extensively covered in the literature [1].

Although out of plane vibrations are of great importance, inplane vibrations can also be important for various applications, e.g. sound transmission, plate systems transmitting inplane forces, or plates subjected to tangential forces, such as the ones produced by the boundary flow of a fluid. Despite this, in plane free vibration analysis of plates has received relatively little attention in the literature. For isotropic plates, in plane free vibration has only recently been studied with some success in [2–5] and in particular, using the dynamic stiffness method [6]. Far less attention has been paid to inplane free vibration of composite plates. A recent contribution to the literature on the subject is by Woodcock et al. [7] where the Rayleigh–Ritz method is used to compute the natural frequencies of a single layer composite square plate for different ply orientations.

For thick composite plates, bending and inplane modes can both occur within the first 10 natural frequencies. It is thus instructive that both of the two motions are studied together. No publication in the literature has so far been identified which deals with both bending and inplane free vibrations of composite plates in an exact manner, particularly including shear deformation and rotatory inertia.

The essential purpose of this two-part paper is not to show in particular, how much difference the effects of shear deformation and rotatory inertia makes to the natural frequencies and mode shapes of a laminated composite plate when using the first order shear deformation theory as opposed to classical plate theory because there are literally dozens of papers in the literature dealing with this subject which have made such assessments [8–18]. It is obviously clear and well known from published literature that the effects could be significant, particularly for thick composite plates, and the importance of the topic becomes even more acute because fibre reinforced composites having low shear moduli are sensitive to the shear deformation effects, unlike isotropic materials. The main purpose of this paper is thus to give a new methodology to deal with the free vibration problems of laminated composite plates using the dynamic stiffness method based on the first order shear deformation theory as a more accurate and efficient alternative to the commonly used finite element method (FEM) [19] rather than pin-pointing the difference in results when using classical plate theory (CPT).

In Part I [20] of this two part paper, a more efficient method to investigate the free vibration behaviour of composite plates has been presented. This method is the dynamic stiffness method (DSM) which has been developed for laminated plates based on the first order shear deformation theory for both bending and inplane vibration. The theory has been implemented in a computer program called DySAP, written in MATLAB enabling the computation of exact natural

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frequencies and mode shapes of complex structures modelled as composite plate assemblies.

In this second part, attention is focused on results. First the dynamic stiffness elements presented in [20] are validated against exact results available in the literature. This has been possible for simple square or rectangular composite plates. In order to demonstrate the efficiency and accuracy of the present method, the results are also compared to approximate solutions obtained by the commercially available FEM package NASTRAN. For bending vibration, the results are discussed in Section 2.1, whereas for inplane vibration they are discussed in Section 2.2. For thick plates showing both bending and inplane modes within the first 10 natural frequencies, Carrera's Unified Formulation (CUF) has been used [8–12] for comparison purposes since it provides analytical results in contrast to FE based numerical ones.

The developed theory has been further used to compute the exact natural frequencies and mode shapes of stringer panels (Section 3) so as to demonstrate the application of the theory to real structures. The exact results of such structures have never been reported before in the literature. The results from the present theory are also compared with approximate results obtained using NASTRAN. Finally, the efficiency, accuracy and versatility of the DSM when studying the free vibration behaviour of real composite structures are demonstrated.

## 2. Validation of results for simple composite plates

### 2.1. Free vibration in bending

The out of plane (or bending) free vibration analysis of a composite square plate is first carried out to validate the theory. The relative material properties, plate dimensions, and laminate lay-up are as follows: $E_1/E_2 = 40$, $h/a = 0.1$, $a = b = 1m$, $G_{12} = G_{13} = 0.6E_2$, $G_{23} = 0.5E_2$, $\nu_{12} = 0.25$, $k = 5/6$ and lay-up = [0/90/0].

The first 6 natural frequencies of the plate are shown in Table 1 for different boundary conditions (S simply supported, C clamped, F free). The dimensionless natural frequency parameter ($\omega^* = \omega a^2/h\sqrt{\rho/E_2}$) together with the corresponding semi-wavelength numbers ($m$ and $n$) are given and the results are compared with those available in the literature [21,22] for validation purposes. Approximate results obtained using CQUAD4 NASTRAN elements are also shown. The comparative exact results from the literature [21,22] are based on the so-called classical method (CM) which uses a Navier's or Levy's type solution and imposes zero or non-zero boundary conditions for displacements and/or forces. This approach can only be used to solve simple plates and cannot be easily extended to structures with complex geometry unlike the DSM. In Table 1, it can be seen that there is total agreement between the solution obtained using DySAP and the exact results reported in the literature [21,22] in which only the first three natural frequencies are quoted. It can also be observed in Table 1 that NASTRAN consistently produces conservative estimate of the natural frequencies with errors ranging from −0.3% to −6.2% on the first 6 natural frequencies. Understandably, the error would increase for higher natural frequencies. This can be attributed to the fact that the FEM gives an approximate solution for the total elastic energy and since a higher energy is associated with higher modes, a greater error is expected. In Fig. 1 some representative modes obtained by using DySAP are compared with those obtained from the FEM analysis. It can be seen that there is excellent agreement between the FE results and the DySAP ones. It should be noted that DySAP results are mesh independent and the mesh used in Fig. 1 is merely a plotting grid for convenience.

Further validation cases can be found in Table 2 where a simply supported square plate is used as an example. The material properties and dimensions of the plate are: $G_{12} = G_{13} = 0.6E_2$, $G_{23} = 0.5E_2$, $\nu_{12} = 0.25$, $k = 5/6$, lay-up = [0/90/90/0], $h/a = 0.2$, $a = b = 1m$. Separate analyses have been carried out for different

**Table 1**

First 6 dimensionless bending frequencies $\omega^* = \omega a^2/h\sqrt{\rho/E_2}$ for a square composite plate with different boundary conditions. Exact results from [21,22]. FEM results by NASTRAN (mesh 50 × 50 CQUAD4 elements). DySAP results are mesh independent. Some of the frequencies have been either not shown (n/s) or missed (m) in the published literature.

| Mode | SSSS | | | | SSSC | | | |
|---|---|---|---|---|---|---|---|---|
| | Exact [21] | DySAP | | FEM | Exact [21] | DySAP | | FEM |
| | $\omega^*$ | $m$ $n$ | $\omega^*$ | $\omega^*$ (error %) | $\omega^*$ | $m$ $n$ | $\omega^*$ | $\omega^*$ (error %) |
| 1 | 14.766 | 1 1 | 14.766 | 14.716 (−0.3) | 17.175 | 1 1 | 17.175 | 17.059 (−0.7) |
| 2 | 22.158 | 2 1 | 22.158 | 21.718 (−2.0) | 23.677 | 2 1 | 23.676 | 23.241 (−1.8) |
| 3 | 36.900 | 3 1 | 36.900 | 34.945 (−5.3) | 37.720 | 3 1 | 37.720 | 35.814 (−5.1) |
| 4 | n/s | 1 2 | 37.380 | 37.072 (−0.8) | n/s | 1 2 | 38.326 | 37.976 (−0.9) |
| 5 | n/s | 2 2 | 41.158 | 40.728 (−1.0) | n/s | 2 2 | 41.942 | 41.495 (−1.1) |
| 6 | n/s | 3 2 | 50.896 | 49.268 (−3.2) | n/s | 3 2 | 51.461 | 49.853 (−3.1) |
| | SCSC | | | | SFSF | | | |
| | Exact [21] | DySAP | | FEM | Exact [22] | DySAP | | FEM |
| | $\omega^*$ | $m$ $n$ | $\omega^*$ | $\omega^*$ (error %) | $\omega^*$ | $m$ $n$ | $\omega^*$ | $\omega^*$ (error %) |
| 1 | 19.669 | 1 1 | 19.669 | 19.490 (−0.9) | 4.343 | 1 1 | 4.343 | 4.302 (−0.9) |
| 2 | 25.349 | 2 1 | 25.349 | 24.915 (−1.7) | missed | 1 2 | 6.262 | 6.201 (−1.0) |
| 3 | 38.650 | 3 1 | 38.650 | 36.795 (−4.8) | 16.212 | 2 1 | 16.212 | 15.675 (−3.3) |
| 4 | n/s | 1 2 | 39.082 | 38.700 (−1.0) | missed | 2 2 | 18.175 | 17.619 (−3.1) |
| 5 | n/s | 2 2 | 42.585 | 42.125 (−1.1) | missed | 1 3 | 30.340 | 30.307 (−0.1) |
| 6 | n/s | 3 2 | 51.938 | 50.347 (−3.1) | 33.186 | 3 1 | 33.186 | 31.121 (−6.2) |
| | SSSF | | | | SFSC | | | |
| | Exact [22] | DySAP | | FEM | Exact [22] | DySAP | | FEM |
| | $\omega^*$ | $m$ $n$ | $\omega^*$ | $\omega^*$ (error %) | $\omega^*$ | $m$ $n$ | $\omega^*$ | $\omega^*$ (error %) |
| 1 | 4.914 | 1 1 | 4.914 | 4.869 (−0.9) | 7.331 | 1 1 | 7.331 | 7.296 (−0.5) |
| 2 | 16.742 | 2 1 | 16.742 | 16.200 (−3.2) | 17.558 | 2 1 | 17.557 | 17.045 (−2.9) |
| 3 | missed | 1 2 | 21.670 | 21.627 (−0.2) | missed | 1 2 | 23.172 | 23.066 (−0.5) |
| 4 | missed | 2 2 | 27.881 | 27.499 (−1.4) | missed | 2 2 | 28.961 | 28.566 (−1.4) |
| 5 | 33.644 | 3 1 | 33.644 | 31.579 (−6.1) | 34.019 | 3 1 | 34.019 | 31.981 (−6.0) |
| 6 | n/s | 3 2 | 41.057 | 39.220 (−4.5) | n/s | 3 2 | 41.721 | 39.918 (−4.3) |