Computers and Structures 136 (2014) 90–97

Contents lists available at ScienceDirect

Computers and Structures

journal homepage: www.elsevier.com/locate/compstruc

Torsional analysis of thin-walled beams of open sections by the direct boundary element method

J.M.F. Cruz, A.V. Mendonça *

Dept. of Civil and Environmental Engineering, Federal University of Paraiba, CEP58059-900 João Pessoa, PB, Brazil

ARTICLE INFO

Article history:
Received 22 May 2013
Accepted 8 February 2014
Available online 5 March 2014

Keywords:
BEM
Nonuniform torsion
Integral equations
Fundamental solution

ABSTRACT

This article presents a direct boundary element formulation for static torsional analysis of open section thin-walled members. All mathematical representations of the problem such as integral equations, fundamental solutions and algebraic systems are established in terms of typical quantities (twisting moment, bimoment, angle of twist and its rate) of Vlasov's nonuniform torsion theory. Numerical results are presented for different cases of torque loadings and boundary conditions.

1. Introduction

Torsional analysis is one of the available ways generally used to investigate strain and stress fields (especially shear ones) over each cross-section of a one-dimensional structural member. If external torque is uniform and axial displacements (due to warping section) are assumed varying over the cross-section but remain constant along the member length, a pure shear stress field associated with uniform torsion state arises according to Saint-Venant's theory. On the contrary, if the member is not free to warp additional shear and normal stresses are induced, known as nonuniform torsion, so that warping's description is generally divided into two types: primary and secondary warping. The former is associated with warping of mid-plane of the cross-section and it is assumed constant across the wall thickness. Primary warping have been incorporated by several torsion models for thin-walled beams such as open-section Vlasov's theory [1], and closed-section torsion approached by Benscoter [2] and Nowinski [3]. The secondary warping is associated with warping of the section across wall thickness and its detailed discussions can be found elsewhere for instance [4–6].

For nonuniform torsion problems under primary warping, analytical solutions for thin-walled beams can be found in [7–9] while numerical solutions basically have been developed based on domain methods – such as finite elements [10–18] – and more recently on boundary element method, whose major contributions to nonuniform torsion analysis have been supported mainly by Sapountzakis and co-authors' works [19–21].

The main strategy is to transform the real problem into an analog problem, this is so-called the analog problem method. Then, integral equations are established to the analog problem using its own fundamental solutions. It should be noted that the analog BEM formulation is an effective approach to solve nonuniform torsion of arbitrary cross-section bar (open or closed), but its mathematical structures do not incorporate directly typical quantities of thin-walled problem (such as bimoment, twisting moment, etc.) into its fundamental solutions, integral and algebraic representations. Hence, the analog BEM can be classified as an indirect BEM formulation.

Unlike the analog BEM solution by Sapountzakis and co-workers [19,20] for a nonuniform torsion of an arbitrary section, in this paper both real and fundamental quantities in integral and algebraic equations are retained, consisting a direct formulation of BEM for Vlasov's nonuniform torsion of open section. To the best author's knowledge, all mathematical representations (integral equations, fundamental solutions, and algebraic systems) presented here are new.

2. Real and fundamental problems

The hypotheses of classical theory of thin-walled beams with open cross section are given by Vlasov [1]: (a) the cross section is rigid in its own plane; (b) the shear strain in the middle surface is neglected. Some of the Vlasov's theory corollaries are: primary warping function is taken as sectorial area coordinate and secondary warping is neglected.

* Corresponding author. Tel.: +55 8396601253.
E-mail address: mendonca@ct.ufpb.br (A.V. Mendonça).

http://dx.doi.org/10.1016/j.compstruc.2014.02.002
0045-7949/

Being a member of thin-walled open section with length L subjected to a distributed torque $m = m(x)$, the governing equation for torsional analysis of the real problem in terms of higher derivatives of angle of twist $\theta(x)$ is given as follows:

$$EI_\omega d^4\theta(x)/dx^4 - GI_t d^2\theta(x)/dx^2 = m(x) \quad (1)$$

where E is the Young's modulus and G the shear modulus. I_t and I_ω are respectively, the Saint Venant's torsion constant and the warping torsion constant (sectorial moment of inertia). ω is the sectorial coordinate.

The constitutive relations for twisting moment and bimoment are respectively given by:

$$T(x) = T_l(x) + T_\omega(x) \quad (2a)$$

$$B(x) = -EI_\omega d^2\theta(x)/dx^2 \quad (2b)$$

where Saint-Venant twisting moment is $T_l(x) = GI_t d\theta(x)/dx$ and warping moment is

$$T_\omega(x) = -EI_\omega d^3\theta(x)/dx^3. \quad (2c)$$

By analogy to the real problem Eq. (1), the balance of the fundamental problem of non-uniform torsion can be expressed as:

$$d^4\theta^*(x,\hat{x})/dx^4 - \lambda_e^2 d^2\theta^*(x,\hat{x})/dx^2 = m^*(x,\hat{x})/(EI_\omega) \quad (3)$$

where $\lambda_e = \sqrt{GI_t/EI_\omega}$, $m^*(x,\hat{x}) = \delta(x,\hat{x})$, and $\delta(x,\hat{x})$ is the Dirac delta.

The constitutive relations of fundamental problem are analogous to Eq. (2), resulting in:

$$T^*(x,\hat{x}) = GI_t d\theta^*(x,\hat{x})/dx - EI_\omega d^3\theta^*(x,\hat{x})/dx^3 \quad (4a)$$

$$B^*(x,\hat{x}) = -EI_\omega d^2\theta^*(x,\hat{x})/dx^2 \quad (4b)$$

where $T^*(x,\hat{x})$ and $B^*(x,\hat{x})$ are fundamental twisting moment and bimoment. x and $\hat{x}$ are the field and source points.

Taking the solution of Eq. (3) in the form:

$$\theta^*(x,\hat{x}) = A + Br + C\cosh(\lambda_e r) + D\sinh(\lambda_e r) \quad (5)$$

where $r = |x - \hat{x}|$, one obtains its first two derivations in x:

$$\psi^*(x,\hat{x}) = d\theta^*(x,\hat{x})/dx = [B + C\lambda_e\sinh(\lambda_e r) + D\lambda_e\cosh(\lambda_e r)]sgn(x-\hat{x}) \quad (6)$$

$$d^2\theta^*(x,\hat{x})/dx^2 = C\lambda_e^2\cosh(\lambda_e r) + D\lambda_e^2\sinh(\lambda_e r) + 2[B + C\lambda_e\sinh(\lambda_e r) + D\lambda_e\cosh(\lambda_e r)]\delta(x,\hat{x}) \quad (7)$$

Differentiation of Eq. (7) implies on derivatives of Dirac delta, which can be avoided if the following relationship is set:

$$B + D\lambda_e = 0 \quad (8)$$

With help of Eq. (8), the second derivative of Eq. (7) can be re-written as:

$$d^2\theta^*(x,\hat{x})/dx^2 = C\lambda_e^2\cosh(\lambda_e r) + D\lambda_e^2\sinh(\lambda_e r) \quad (9)$$

The third and fourth derivatives of $\theta^*(x,\hat{x})$ in Eq. (5) can be obtained from first and second derivatives of Eq. (9), resulting:

$$d^3\theta^*(x,\hat{x})/dx^3 = \{C\lambda_e^3\sinh(\lambda_e r) + D\lambda_e^3\cosh(\lambda_e r)\}sgn(x-\hat{x}) \quad (10)$$

$$d^4\theta^*(x,\hat{x})/dx^4 = C\lambda_e^4\cosh(\lambda_e r) + D\lambda_e^4\sinh(\lambda_e r) + 2\left[C\lambda_e^3\sinh(\lambda_e r) + D\lambda_e^3\cosh(\lambda_e r)\right]\delta(x,\hat{x}) \quad (11)$$

Substituting the Eqs. (9) and (11) into governing equation of fundamental problem Eq. (3), yields:

$$2\left[C\lambda_e^3\sinh(\lambda_e r) + D\lambda_e^3\cosh(\lambda_e r)\right]\delta(x,\hat{x}) = \delta(x,\hat{x})/(EI_\omega) \quad (12)$$

The identity in Eq. (12) is always true if:

$$D = 1/(2\lambda_e^3 EI_\omega) = 1/(2\lambda_e GI_t) \quad (13)$$

Substituting Eq. (13) into Eq. (8) yields to:

$$B = -1/(2\lambda_e^2 EI_\omega) = -1/(2GI_t) \quad (14)$$

The fundamental solution of angle of twist is obtained substituting Eqs. (13) and (14) into Eq. (5):

$$\theta^*(x,\hat{x}) = A - r/(2GI_t) + C\cosh(\lambda_e r) + \sinh(\lambda_e r)/(2\lambda_e GI_t) \quad (15)$$

The constants A and C are arbitrary. If they are set $A = C = 0$, then Eq. (15) can be simplified to:

$$\theta^*(x,\hat{x}) = [-\lambda_e r + \sinh(\lambda_e r)]/(2\lambda_e GI_t) \quad (16)$$

Other fundamental fields of interest are:

$$\begin{aligned}
&\psi^*(x,\hat{x}) = d\theta^*(x,\hat{x})/dx = sgn(r)\{-1 + \cosh(\lambda_e r)\}/(2GI_t)\\
&B^*(x,\hat{x}) = -EI_\omega d^2\theta/dx^2 = -\sinh(\lambda_e r)/(2\lambda_e)\\
&T^*(x,\hat{x}) = -EI_\omega d^3\theta^*(x,\hat{x})/dx^3 + GI_t d\theta^*(x,\hat{x})/dx = -sgn(r)/2\\
&d\theta^*(x,\hat{x})/d\hat{x} = \theta^*_{,\hat{x}}(x,\hat{x}) = -sgn(r)\{-1 + \cosh(\lambda_e r)\}/(2GI_t)\\
&d\psi^*(x,\hat{x})/d\hat{x} = \psi^*_{,\hat{x}}(x,\hat{x}) = -\lambda_e\sinh(\lambda_e r)/(2GI_t)\\
&dB^*(x,\hat{x})/d\hat{x} = B^*_{,\hat{x}}(x,\hat{x}) = sgn(r)\{\cosh(\lambda_e r)\}/2\\
&dT^*(x,\hat{x})/d\hat{x} = T^*_{,\hat{x}}(x,\hat{x}) = \delta(x,\hat{x})
\end{aligned} \quad (17a\text{-}g)$$

3. Integral and algebraic equations

If Eq. (1) is weighted by fundamental solution Eq. (16), the method of weighted residuals states:

$$\int_0^L [EI_\omega d^4\theta(x)/dx^4 - GI_t d^2\theta(x)/dx^2 - m(x)]\theta^*(x,\hat{x})dx = 0 \quad (18)$$

After four integrations by parts of Eq. (18) and then with help of Eq. (2a–b), Eq. (3) and Eq. (4a–b) and the property of the Dirac delta, one obtains the integral equations for the angle of twist:

$$\theta(\hat{x}) + [T^*(x,\hat{x})\theta(x)]_0^L - [B^*(x,\hat{x})\psi(x)]_0^L = [T(x)\theta^*(x,\hat{x})]_0^L - [B(x)\psi^*(x,\hat{x})]_0^L + \int_0^L m(x)\theta^*(x,\hat{x})dx \quad (19)$$

The torsion problem requires two unknowns at boundary to be determined. Hence, an additional equation is necessary to be established in order to get the problem solvable. Then, this remaining equation can be associated with the derivative of Eq. (19) at source point $\psi(\hat{x}) = d\theta(\hat{x})/d\hat{x}$, yielding to integral equation for the rate of angle of twist:

$$\psi(\hat{x}) + \left[T^*_{,\hat{x}}(x,\hat{x})\theta(x)\right]_0^L - \left[B^*_{,\hat{x}}(x,\hat{x})\psi(x)\right]_0^L = \left[T(x)\theta^*_{,\hat{x}}(x,\hat{x})\right]_0^L - \left[B(x)\psi^*_{,\hat{x}}(x,\hat{x})\right]_0^L + \int_0^L m(x)\psi^*(x,\hat{x})dx \quad (20)$$

The governing equation of the beam Eq. (1) is defined on one dimensional domain (1-D). After application of both integration by parts and with the help fundamental solutions, the quantities at domain points given in Eqs. (19) and (20) are written in terms of boundary variables only. Hence the problem dimensionality to be solved is reduced by one, that is, 1-D description of beam domain is replaced by a 0-D mathematical representation associated with the end points (boundary) of the beam.

In BEM formulations, the transformation of integral equations into an algebraic representation is done by discretization of the problem using boundary elements and evaluation of the integrals. For a beam problem, the boundary elements are the end points of the beam.

The algebraic representation in terms of boundary quantities for displacements (see Fig.1(a)) and for forces (see Fig.1(b)) can be