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Analysis of moderately thin-walled beam cross-sections by cubic isoparametric elements

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ABSTRACT

In technical beam theory the six equilibrium states associated with homogeneous tension, bending, shear and torsion are treated as individual load cases. This enables the formulation of weak form equations governing the warping from shear and torsion. These weak form equations are solved numerically by introducing a cubic-linear two-dimensional isoparametric element. The cubic interpolation of this element accurately represents quadratic shear stress variations along cross-section walls, and thus moderately thin-walled cross-sections are effectively discretized by these elements. The ability of this element to represent curved geometries, and to accurately determine cross-section parameters and shear stress distributions is demonstrated.

1. Introduction

Beam theories are typically based on a separation of the kinematics and statics into lengthwise and a cross-section behavior. The level of detail may vary, depending on the intended application from very simple representations typical of building structures to quite sophisticated theories necessary when describing the interface behavior of composite beams. For civil engineering applications thin-walled structural elements are often used in steel structures or concrete bridges, where the influence of cross-section warping and distortion from eccentric loading into combined flexure and torsion [1] has received considerable attention. Beam elements that take these effects into account have been formulated in for example [2–4], with associated stiffness parameters derived in [5]. In the helicopter and wind energy communities the main focus has been on the formulation of beam theories and cross-sectional analysis tools that are able to cope with advanced anisotropic material distributions in composite beams. In particular, several dedicated numerical codes are available for the analysis of beam cross-sections. The most widely recognized concept is that of VABS, presented in for example Hodges et al. [6–9]. The basis of VABS is the formulation of 3D equations of nonlinear elasticity with large deformation strain measures. These coupled equations are separated into a 1D problem in the longitudinal direction and a 2D problem with respect to the plane orthogonal to the longitudinal direction by an asymptotic formulation, where the truncation is based on the assumption that the scale of the 2D domain is small relative to the other length scales of the full elasticity problem. In the 2D problem the warping functions are represented by isoparametric elements [6,10]. A clear advantage of VABS is the versatility of the concept, where for instance arbitrary fiber orientations, initial curvature and pre-twist and shear flexibility may be incorporated via the 3D elasticity formulation [7]. Also, VABS has been used for open section thin-walled analysis in [8], and for modeling of composite rotor blades in [9]. A state-of-the-art description of VABS can be found in the user's manual [10]. An alternative approach based on a virtual work formulation was presented by Giavotto et al. [11], where the displacement fields are discretized by planar finite elements. This approach has recently been used to develop the cross-section code BECAS [12,13] for the analysis of wind turbine blades. Furthermore, the cross-section analysis strategy in [11] has been combined with aeroelastic and structural multi-body codes to establish a global optimization framework for the design of future wind turbines [14,15]. For the analysis of advanced beam structures the accurate estimation of torsion and shear stiffness is of significant interest, and isoparametric finite elements of different order have been used in [16–20] to solve the torsion and/or shear problems for various types of cross-sections.

The aim of the present paper is the consistent formulation of a beam theory and the subsequent numerical derivation of the associated cross-section parameters by using suitable isoparametric elements for the discretization of the solution domain. The paper initially gives a description of a beam theory based on the homogeneous load cases of tension, bending, shear and torsion [21–23], which is equivalent to the formulation in terms of complementary energy [24]. It corresponds closely to technical beam theory, but

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includes shear effects, and accounts for the shear stress distribution in the states of homogeneous shear and torsion. The key point is the derivation of partial differential equations for the shear and torsion problems, and extraction of the governing expressions for the associated beam stiffness parameters. The beam cross-section considered in this paper is general and the material assumption is transverse anisotropy. The background theory is similar to that used in [25–29] for analysis of pre-twisted beams, but in the present paper it is extended to include a detailed variational formulation of the problem of the cross-section stiffness parameters. In thin-walled beams the out-of-plane warping deformations due to torsion may be significant, and the additional stiffness contribution obtained by restraining warping is part of the Vlasov beam theory, see [30–33]. As the present theory is established for homogeneous loading cases it contains the out-of-plane warping due to torsion explicitly in the kinematic description, and therefore directly provides the full set of cross-section parameters for thin-walled Vlasov beam theory.

The theory is implemented in the MATLAB-based finite element code BeamSec, which determines cross-section parameters based on a 2D formulation in terms of isoparametric elements. The details of the implementation are presented in A. The numerical implementation is formulated in terms of appropriate warping functions for shear and torsion. An important feature of the applied finite elements is the use of cubic interpolation in the direction of the shear stress in order to satisfy the basic variation of the stresses in for example a rectangular section in a state of homogeneous shear. In the wall thickness direction the interpolation is linear, which limits the accuracy of the present element to cross-sections with moderate wall thickness. The examples demonstrate that the finite element with cubic interpolation can accurately represent curved geometries as long as the intermediate element nodes are located at the one- and two-third points with respect to the length of the (curved) element side. Thus, the ability of this cubic-linear element to accurately represent shear stress variations and curved geometries implies that cross-sections may be discretized by few elements and that cross-section properties can be obtained with limited computational effort.

2. Kinematic assumptions

Consider a beam with longitudinal coordinate z and cross-section coordinates $\mathbf{x} = [x_1, x_2]^T$, as shown in Fig. 1, which supports the deformation states of extension, bending, shear and torsion. The displacements and deformations are associated with the coordinates of the elastic center (c_1, c_2) and the shear center (a_1, a_2). This is illustrated in Fig. 1, where transverse translation (ξ_1, ξ_2) and twist φ are referred to the shear center, while axial extension ζ and inclination (rotation) of the cross-section (η_1, η_2) are referred to the elastic center. This leads to the transverse displacement components

$$u_\alpha(\mathbf{x}, z) = \xi_\alpha(z) - e_{\alpha\beta}(x_\beta - a_\beta)\varphi(z) \tag{1}$$

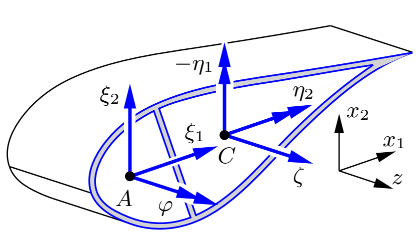

Fig. 1. Cross-section with elastic center C and shear center A.

where Greek subscripts $\alpha, \beta = 1, 2$ represent the two in-plane directions. In (1) $e_{\alpha\beta}$ is the two-dimensional permutation tensor and summation over repeated (Greek) subscripts is implied in the following. The axial displacement of the beam is represented in the form

$$w(\mathbf{x}, z) = \zeta(z) + (x_\beta - c_\beta)\eta_\beta(z) + \psi_\beta(\mathbf{x})(\xi'_\beta(z) + \eta_\beta(z)) - \omega(\mathbf{x})\varphi'(z) \tag{2}$$

where $()' = d()/dz$ represents the gradient of the kinematic and static components with respect to the longitudinal axis of the beam. In the expression for the axial displacement $\psi_\beta(\mathbf{x})$ and $\omega(\mathbf{x})$ are functions of the cross-section coordinates $\mathbf{x}$ describing the warping due to shear and torsion, respectively. In Fig. 1 the double-tip arrows represent the cross-section rotations, which is the reason for the interchange of the components η_1, η_2 and the change in sign on the inclination η_1.

The warping is assumed to be homogeneous, and thus does not account for the possibility of constrained warping in zones at the ends of the beam. Therefore, the axial strain is defined as

$$\varepsilon(\mathbf{x}, z) = \frac{\partial w}{\partial z} \simeq \zeta'(z) + (x_\beta - c_\beta)\eta'_\beta(z) \tag{3}$$

This representation with contributions only from pure extension and bending is justified for beam deformations from the homogeneous equilibrium states considered in Section 3. Furthermore, it implies that restrained warping stiffness accounts for elastic energy accumulated by the specific warping functions associated with free out-of-plane deformation. The shear strains are defined by their full expressions as

$$\begin{aligned}\gamma_\alpha(\mathbf{x}, z) &= w_{,\alpha} + u'_\alpha \\ &= \left[\delta_{\alpha\beta} + \psi_{\beta,\alpha}(\mathbf{x})\right](\xi'_\beta(z) + \eta_\beta(z)) \\ &\quad - \left[e_{\alpha\beta}(x_\beta - a_\beta) + \omega_{,\alpha}(\mathbf{x})\right]\varphi'(z)\end{aligned} \tag{4}$$

where $()_{,\alpha} = \partial()/\partial x_\alpha$ are the in-plane gradients and $\delta_{\alpha\beta}$ is Kroneckers delta. The first set of square brackets in (4) contains the cross-section distribution of the shear strains from a homogeneous state of shear in the beam, while the second set of square brackets contains the distribution of shear strains from homogeneous torsion. In the present theory cross-section deformation, and thereby the in-plane strain components, are neglected. This is a reasonable assumption when calculating the overall beam characteristics, but excludes direct analysis of cross-section distortion due to e.g. inhomogeneous material distribution, and the associated development of stresses between different parts of the section due to this material variation.

In the present approximate beam theory the main stress components are the axial stress $\sigma(\mathbf{x}, z)$ and the shear stress components $\tau_\alpha(\mathbf{x}, z)$. Although the shear stress components for thin-walled cross-sections will be aligned mainly along the individual flanges, it is advantageous to consider the two shear stress components as constituting the vector solution to a two-dimensional boundary value problem. The elastic properties are assumed transversely isotropic with axial stress

$$\sigma(\mathbf{x}, z) = E(\mathbf{x})\,\varepsilon(\mathbf{x}, z) \tag{5}$$

and shear stress components

$$\tau_\alpha(\mathbf{x}, z) = G(\mathbf{x})\,\gamma_\alpha(\mathbf{x}, z) \tag{6}$$

introducing the elastic modulus $E(\mathbf{X})$ and the shear modulus $G(\mathbf{x})$, respectively.