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On the stability of the finite element immersed boundary method ☆

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Abstract

The immersed boundary (IB) method is a mathematical formulation for fluid–structure interaction problems, where immersed incompressible visco-elastic bodies or boundaries interact with an incompressible fluid.

The original numerical scheme associated to the IB method requires a smoothed approximation of the Dirac delta distribution to link the moving Lagrangian domain with the fixed Eulerian one.

We present a stability analysis of the finite element immersed boundary method, where the Dirac delta distribution is treated variationally, in a generalized visco-elastic framework and for two different time-stepping schemes.

Keywords: Immersed boundary method; Finite element method; Numerical stability; CFL condition

1. Introduction

One of the main difficulties that arises when dealing with visco-elasticity and fluid–structure interaction problems is the fact that fluids and elastic materials have a different "natural" framework.

The usual way of characterizing a fluid motion is the Eulerian framework, where the system is described using the velocity and pressure fields. On the other hand, when dealing with elasticity, it is customary to express the stress as a function of the displacements of the material particles from their reference, or Lagrangian, position, which is not directly available in the Eulerian formulation.

The immersed boundary method gives one way to link the two frameworks together and deploy the strengths of both formulations at the same time. The original IB formulation (see [25] for an introduction on the subject) was intended to simplify the study of the interaction between thin membranes undergoing large deformations and fluids described by the Navier–Stokes equations, by means of an approximation of the Dirac delta distribution, which was used as an interpolation Kernel between the two frameworks.

A finite element formulation of the problem was first introduced in [21] and later developed in [32,34], where the discretization of the fluid is done via the finite element method and the passage from the Eulerian to the Lagrangian domain is done via the Reproducing Kernel Particle Method, to provide an approximation of the Dirac delta distribution suitable for the finite element method.

A variational approach to the problem was introduced in [2,3], where the Dirac delta distribution is no longer needed as it is treated variationally through its action on the test functions. The variational approach translates naturally in a finite element formulation of the IB method which was further developed in [4].

The original discretization of the interaction equations, as proposed in [25], preserves mass, momentum, angular momentum, torque and power, ensuring that in the conversion between the two frameworks no spurious creation or destruction of mass, momentum or energy is induced by the numerical approximation. The introduction of the time discretization however disrupts these conservation properties.

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In [31], the authors present a comparison between three different time approximation schemes, highlighting the difficulties related to the nonlinearity of the coupled problem. A first attempt to analyze the stability properties of these approximation schemes was introduced in [30] where the authors present a stability analysis based on the study of the modes of oscillation of a single straightened one-dimensional fiber immersed in a two-dimensional fluid. The effect of these modes of oscillation on the time-stepping schemes was further analyzed by the same authors in [29].

In [5,7,8,6] the authors present a stability analysis that takes advantage of the finite element formulation of the problem and of the natural energy estimates inherited by the variational analysis of the coupled system. The idea is based on the requirement that the energy of the system decreases at each time step, providing an effective CFL condition for the coupled problem to remain stable with respect to the fluid–structure interaction characteristics.

The extension to more general fluid–structure interaction problems using the formulation derived in [25] was limited to anisotropic elasticity due to the lack of a term in the formulation that takes care of the continuity of the stress between the solid body and the fluid.

In [10], the authors recognized this problem and proposed a derivation of the IB method based on classical hyper-elasticity theory (see, for example, [15]) where the missing term was found to be of the same character of the original singular term introduced for the study of thin membranes.

In this paper, we review the finite element immersed boundary (FEIB) method as introduced in [2–4,10] and we generalize the results presented in [8] to take into account general hyper-elastic materials, both in the thin case of co-dimension one structures interacting with two- or three-dimensional fluids as well as in the more general framework of two- or three-dimensional structures interacting with two- or three-dimensional fluids.

Sections 2 and 3 present briefly the IB method and the hyper-elastic models that will be used. Section 4 present the variational and finite element formulation of the IB method, while in Sections 5 and 6 we present the time discretization and the generalized stability analysis for the fully discrete problem.

Sections 7 and 8 present some numerical experiments and conclusions.

2. The immersed boundary method

The formulation of the immersed boundary method that we use in the following stability analysis is the one that was derived in [10]. Some simplifying assumptions are made that render the analysis more comprehensible, however a generalization is possible following the same lines presented in this paper.

We study a region Ω of $\Re^d$ containing a fluid and an immersed solid material. In particular we are interested in the interaction between the two when the fluid is described by the Navier–Stokes equations and the solid is described by a viscous hyper-elastic model.

We identify the material particles contained in Ω with the set $\omega \subseteq \Re^d$ of their position in a reference configuration and we suppose that the subset $\mathscr{B} \subseteq \omega \subseteq \Re^d$ identifies the elastic body. The coordinate system in this Lagrangian framework is given by the vector variable $\mathbf{s}$. When $\mathbf{s} \in \mathscr{B}$ it marks a solid particle, while if $\mathbf{s} \in \omega \setminus \mathscr{B}$ it identifies a fluid particle.

We fix a coordinate system for the physical space where the position variable $\mathbf{x}$ is used to determine a fixed point in space. For simplicity we assume that during the entire motion all the material particles (both solid and fluid) remain contained in the region $\Omega \subset \Re^d$.

The relationship between the two different frameworks is given by the mappings:

$$\begin{aligned} \mathbf{X}: &\quad \omega \times [0,T] \mapsto \Omega \supseteq \mathscr{B}_t, \\ \mathbf{q}: &\quad \Omega \times [0,T] \mapsto \omega \supseteq \mathscr{B}, \end{aligned} \tag{1}$$

representing in the first case the trajectory of a material particle (when $\mathbf{s}$ is fixed) or the mapping between the reference configuration and the current one (when t is fixed) and the inverse mapping that associates with each point $\mathbf{x} \in \Omega$ the material particle that happens to be there (as in Fig. 1). The following identities hold:

$$\mathbf{x} = \mathbf{X}(\mathbf{q}(\mathbf{x},t),t), \quad \mathbf{s} = \mathbf{q}(\mathbf{X}(\mathbf{s},t),t) \tag{2}$$

and we assume that at any given time the mapping $\mathbf{X}(\mathbf{s},t)$ is invertible, which implies that the deformation gradient

$$\mathbb{F}_{\alpha i} := (\nabla_s \mathbf{X}(\mathbf{s},t))_{\alpha i} = X_{\alpha,i}(\mathbf{s},t) = \frac{\partial X_\alpha(\mathbf{s},t)}{\partial s_i} \tag{3}$$

has a non-zero determinant. We assume it to be positive at time $t = 0$, and therefore at any subsequent time, i.e.

$$|\mathbb{F}| = \det \mathbb{F} > 0. \tag{4}$$

Functions having $\mathbf{x}$ as the space variable are usually called *spatial* or Eulerian functions, while the ones having $\mathbf{s}$ as the space variable are called *material* or Lagrangian functions.

We define the velocity field $\mathbf{u}$ by

$$\mathbf{u}(\mathbf{x},t) = \frac{\partial \mathbf{X}}{\partial t}(\mathbf{s},t)\bigg|_{\mathbf{s}=\mathbf{q}(\mathbf{x},t)} \tag{5}$$

and its total time derivative by

$$\begin{aligned} \frac{\mathrm{D}\mathbf{u}}{\mathrm{D}t}(\mathbf{x},t) &= \frac{\partial^2 \mathbf{X}}{\partial t^2}(\mathbf{s},t)\bigg|_{\mathbf{s}=\mathbf{q}(\mathbf{x},t)} \\ &= \frac{\partial \mathbf{u}}{\partial t}(\mathbf{x},t) + \mathbf{u}(\mathbf{x},t)\cdot\nabla\mathbf{u}(\mathbf{x},t), \end{aligned} \tag{6}$$

which describe respectively the velocity and the acceleration of the particle that happens to be at the point $\mathbf{x}$ at time t.

To ease the reading, we will try to maintain throughout the paper the following notation: capital letters refer to functions whose domain is the Lagrangian one (e.g. $\mathbf{X}(\mathbf{s},t)$), while lower case letters refer to functions whose