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Local enrichment of finite elements for interface problems

Fausto Cavalli [a], Lucia Gastaldi [b,*]

[a] Dipartimento di Matematica, Università di Milano, Via Saldini 50, 20133 Milano, Italy
[b] DICATAM – Sez. Matematica, Università di Brescia, Via Branze 43, 25123 Brescia, Italy

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ABSTRACT

We consider interface problems for second order elliptic partial differential equations with Dirichlet boundary conditions. It is well known that the finite element discretization may fail to produce solutions converging with optimal rates unless the mesh fits with the discontinuity interface. We introduce a method based on piecewise linear finite elements on a non-fitting grid enriched with a local correction on a sub-grid constructed along the interface. We prove that our method recovers the optimal convergence rates both in H^1 and in L^2 depending on the local regularity of the solution. Several numerical experiments confirm the theoretical results.

1. Introduction

In several physical situations, like heat transfer, fluid dynamics, acoustic waves, electromagnetic phenomena and materials science, problems having discontinuous coefficients across interfaces may arise. In general situations, the position of the interfaces can be either given or can be an unknown of the problem as in the case of phase transition, fluid–structure interaction, heterogeneous structures or free-boundary problems. If the interface is fixed, it might be desirable to construct the mesh in such a way it fits with the interface, in order to achieve the optimal accuracy in each subregion of the domain. In the case of unknown interface as those quoted above, a lot of effort has been made to develop efficient numerical techniques (see for example [1–10] and the references therein). Since, in this kind of problems, the interface is time dependent and the problem is nonlinear, computing the correct position of the interface at each time step could require some iterations to guarantee stability of the scheme. In such case, instead of adapting the mesh to fit the interface at each time step, it could be convenient considering meshes which do not take into account the position of the interface and apply the enrichment method, we are going to present, to recover the optimal accuracy.

From the mathematical viewpoint, the lack of global regularity of the coefficients may affect the global regularity of the solution, even if the interface is smooth. In these situations numerical schemes may fail to provide the optimal convergence rate. On the other hand, if the coefficients are locally regular, then the solution could enjoy local smoothness properties. Therefore, in order to have a finite element solution which converges to the continuous one with an optimal rate, the computational meshes need to be constructed in such a way that the interface is well approximated by the mesh faces. In this case, the meshes are called *fitting*. We refer, for example, to [11–14] for the analysis in the elliptic and parabolic case. However, the generation of a fitting grid could be time consuming in presence of complicated geometries or in the case of time dependent interfaces, which would require the re-meshing at each time step. We mention here that one could proceed by constructing independent meshes in the subregions. In such case one gets fitting meshes which could be *non-matching*, that is grids that do not share the same nodes along the interface between two adjacent regions. For example, one can use a fine mesh on a certain region and a coarser mesh on the adjacent one. As a consequence the finite element space in the first region presents more degrees of freedom along the interface than that defined in the second region. In this situation, modern software packages offer a so-called *glue technology* which allows to interpolate the two fields in order to obtain a continuous solution, see e.g. ADINA (http://www.adina.com/newsgB36.shtml).

Several techniques have been proposed which employ *non-fitting* meshes associated with proper strategies which allow to recover the optimal convergence rate. The *immersed interface methods* proposed in [15,16] combine the advantages of cartesian grids with the construction of suitable local piecewise polynomials which can take into account the jump of the normal derivative. The *partition of unity method* [17] allows to include in the finite element spaces a priori knowledge of the behavior of the solution close to the interface. In the *extended finite element method* [18] an enrichment of the standard finite element spaces is constructed in order to model arbitrary discontinuities of functions or their derivatives. A different approach is followed in the unfitting finite element

* Corresponding author. Tel.: +39 030 371 5736; fax: +39 030 371 5745.
E-mail addresses: fausto.cavalli@unimi.it (F. Cavalli), lucia.gastaldi@ing.unibs.it (L. Gastaldi).

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method proposed in [19] where the approximate solution is allowed to be discontinuous and the correct interface conditions are enforced weakly using Nitsche's method. Moreover, we quote the methods based on *fictitious domains*, see [20,21].

Our approach is based on the *immersed boundary method* proposed in [22,23] for simulating fluid motion around solid objects. In these papers the immersed boundary represents the interface between fluid and solid where Dirichlet boundary conditions are imposed. The main idea of the method is to construct a mesh for the union of the fluid and the solid domains independently of the position of the solid and then to cut the elements crossed by the immersed boundary adding new degrees of freedom. Then the additional unknowns in the solid domain can be computed locally in terms of the new degrees of freedom using the boundary condition. In our case, instead, the *immersed boundary* corresponds to the interface where the coefficients present discontinuities, therefore the value of the solution along it is not known in advance. In order to obtain a numerical solution with optimal convergence rate, we introduce first a finite element discretization of the interface problem on a non-fitting grid, then we enrich the resulting finite element space performing a local refinement of the grid along the interface. We observe that the additional unknowns we have introduced could be expressed by static condensation in term of the unknowns on the non-fitting grid, so that the size of the algebraic system would not increase. Moreover, this procedure does not require the modification of the non-fitting mesh structure keeping the original numbering of the degrees of freedom. This advantage becomes more relevant when it is applied to the case of time dependent problems with moving interfaces, as, for example, the matrices of the resulting linear system can be modified only locally at each time step. The construction of the sub-grid associated with the local enrichment can be done in several ways. Our *local* finite element space is obtained by subdividing the elements crossed by the interface into sub-elements so that edges can intersect the interface only at the vertexes.

Here we present and discuss into details the case of two dimensional problems discretized by piecewise linear finite elements, however the method can be generalized to higher order finite elements. We show that the method recovers the optimal rate of convergence provided the solution is sufficiently smooth in each subregion. The three dimensional case can also be treated using, for example, the geometric considerations of [22] for the construction of the sub-grid.

The paper is organized as follows: in Section 2 we present the problem with the regularity results for the solution; next, the enriched finite element method is introduced and the error estimates are provided in Section 4. The last section reports some numerical experiments confirming the theoretical results together with the numerical behavior of the condition number of the involved matrices. In particular, we exploit numerically the behavior of the condition number of the matrix analyzing what happens when the interface gets closer to the existing non-fitting grid so that the sub-elements could fail to satisfy the minimum angle condition.

2. Elliptic problem with discontinuous coefficients

In order to simplify the presentation, we consider a convex polygonal domain $\Omega \subset \mathbb{R}^2$, divided into two open sets $\Omega_1,\ \Omega_2$ by an interface Γ. The technique we are going to present can be extended to the case of several domains with suitable modifications. We assume that Γ is Lipschitz continuous and that it is composed by a finite number of open arcs each of them of class $\mathcal{C}^2$. For a regular function v defined on Ω, we denote by v_i for $i = 1,2$ its restrictions to Ω_i, that is $v_i = v|_{\Omega_i}$ for $i = 1,2$.

We consider the following elliptic problem with homogeneous Dirichlet boundary conditions

$$\begin{aligned} -\nabla\cdot(\beta\nabla u) &= f \quad \text{in } \Omega,\\ u &= 0 \quad \text{on } \partial\Omega, \end{aligned} \tag{1}$$

where f and β are sufficiently smooth functions on Ω. We assume that $\beta \geqslant \beta_0 > 0$ for some $\beta_0 \in \mathbb{R}$. The case of non homogeneous Dirichlet boundary conditions can be treated as well with standard modifications.

When β presents jump discontinuities across Γ, problem (1) can be rewritten as the following transmission problem:

$$\begin{aligned} &-\nabla\cdot(\beta\nabla u_1) = f && \text{in } \Omega_1,\\ &-\nabla\cdot(\beta\nabla u_2) = f && \text{in } \Omega_2,\\ &u_1 = u_2 && \text{on } \Gamma,\\ &\beta_{|\Omega_1}\frac{\partial u_1}{\partial \boldsymbol{n}_1} + \beta_{|\Omega_2}\frac{\partial u_2}{\partial \boldsymbol{n}_2} = 0 && \text{on } \Gamma,\\ &u_1 = 0 && \text{on } \partial\Omega_1 \setminus \Gamma,\\ &u_2 = 0 && \text{on } \partial\Omega_2 \setminus \Gamma. \end{aligned} \tag{2}$$

Let $f \in L^2(\Omega)$ and $\beta \in L^\infty(\Omega)$. We assume that the restrictions of β to the sub-domains Ω_i are sufficiently smooth, that is $\beta_{|\Omega_i} \in W^{1,\infty}(\Omega_i)$ for $i = 1,2$. The variational formulation of problem (1) is then the following: find $u \in H^1_0(\Omega)$ such that

$$a(u,v) = (f,v) \quad \forall v \in H^1_0(\Omega), \tag{3}$$

where

$$a(u,v) = \int_\Omega \beta\nabla u\nabla v\, d\boldsymbol{x} \quad (f,v) = \int_\Omega f v\, d\boldsymbol{x}. \tag{4}$$

We shall use the following notation for the norms in the Hilbert spaces on $B \subset \mathbb{R}^2$: for all $v \in L^2(B)$ we set

$$\|v\|_{0,B} = \left(\int_B v^2 d\boldsymbol{x}\right)^{1/2}.$$

Next for functions v in the Hilbert space $H^s(B)$ with s integer we define

$$\begin{aligned} |v|_{j,B} &= \|D^j v\|_{0,B} \quad \text{for } 0 \leqslant j \leqslant s,\\ \|v\|_{s,B} &= \left(\sum_{j=0}^{s} |v|^2_{j,B}\right)^{1/2}, \end{aligned} \tag{5}$$

where $D^j v$ stands for the array of the derivatives of v of order j. When $B = \Omega$ and no confusion may arise we drop the subscript Ω.

We introduce the space $H^s(\Omega_1 \cup \Omega_2) = \{v \in H^1(\Omega) : v_i \in H^s(\Omega_i)$ for $i = 1,2\}$ endowed with the norm

$$\|v\|_{s,\Omega_1\cup\Omega_2} = \left(\|v_1\|^2_{s,\Omega_1} + \|v_2\|^2_{s,\Omega_2}\right)^{1/2}.$$

Problem (3) has a unique solution $u \in H^1_0(\Omega)$ such that $u \in \cap H^2(\Omega_1 \cup \Omega_2)$ if Γ belongs to C^2, with the following a priori estimate (see [14,24,25])

$$\|u\|_1 + \|u\|_{2,\Omega_1\cup\Omega_2} \leqslant C\|f\|_0.$$

In the case when Γ does not belong to C^2 but it is only globally Lipschitz continuous, the regularity of the restrictions of u to Ω_1 and Ω_2 might be reduced and $u \in H^s(\Omega_1 \cup \Omega_2)$ with $3/2 < s \leq 2$, see [26, Chapter 2].

To simplify the exposition of the numerical method, we assume that the coefficient β is piecewise constant with

$$\beta(x) = \beta_1 \text{ for } x \in \Omega_1, \quad \beta(x) = \beta_2 \text{ for } x \in \Omega_2 \tag{6}$$

and $\beta_1 < \beta_2$. With this choice it is easy to check that the continuity constant of $a(\cdot,\cdot)$ is given by β_2, and the coercivity constant by