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An edge-based smoothed finite element method softened with a bubble function (bES-FEM) for solid mechanics problems

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ABSTRACT

The edge-based smoothed finite element method (ES-FEM) was recently proposed to improve the performance of linearly triangular finite element models for mechanics problems. Such a good performance is attributed to the right amount softening induced by the edge-based smoothing operation. In this paper, we propose an improved formulation of the ES-FEM so that the bES-FEM can be further softened becoming volumetric locking free and hence works well also incompressible or nearly incompressible problems. The improved formulation uses the usual piecewise linear displacements but is supplemented with a cubic bubble function in triangular elements, which induces further softening to the bilinear form allowing the weakened weak (W^2) procedure to search for a solution satisfying the divergence-free conditions. The smoothed strains are evaluated based on smoothing domains associated with edges of the adjacent elements as in the ES-FEM. The divergence-free condition of the bES-FEM is verified via detailed eigenvalue analyses. Several numerical examples are provided to show the effectiveness and reliability of the present method. We also show numerically that the present element is insensitive to mesh distortion and is superior to the bubble finite element (MINI element) in the incompressible limit.

1. Introduction

Nowadays the finite element method (FEM) has become a very powerful and reliable tools for numerical simulations in science and engineering [1–4]. In many practical engineering applications, lower-order finite elements based on linear triangular elements (FEM-T3) are often preferred due to its computational simplicity, efficiency, less demand on the smoothness of the solution, adaptation to complicated geometry, and easy for adaptive mesh refinements for solutions of desired accuracy. However, the FEM-T3 exhibits a poor performance due to certain inherent drawbacks: (1) overestimation of stiffness matrix [1] especially in the incompressible limit and bending dominated behavior; (2) poor performance with distorted meshes; (3) poor accuracy for stresses. There are many methods to overcome these shortcomings in the literature [1,2]. To solve the volumetric locking problems, numerous studies were performed using the drilling freedoms formulations [5–8], u/p mixed formulations [1–3,9], enhanced assumed strain (EAS) modes [8,10,11], B-bar methods [2], reduced integration stabilizations [12–16], two-field mixed stress elements [4], variational multiscale approaches [17,18], average nodal techniques [19–23], just mention to a few. In addition, alternative approaches [24–28] related to meshfree methods [29–32] were addressed for volumetric locking issues.

In the attempts to improve the performance of lower-order finite elements, substantial efforts have been by Liu's group, aiming at reducing the so-called overly-stiffness of the standard FEM-T3 model. A concept of "softening effects" [29,33] was put forward using strain smoothing operations for functions in H^1 space [25], and the generalized gradient smoothing operation for functions in G^1 space [29]. A family of so-called smoothed finite element methods (S-FEMs) has been formulated and desired softening effects were observed [33]. The essential idea in the S-FEMs is to reconstruct the compatible strain field in finite element settings using the strain smoothing technique [25]. In the S-FEMs, the reconstructed strain field is obtained over various sub-domains called smoothing domains created on cells, nodes, edges or faces of the background mesh, and the art of the S-FEM is the innovative design of the smoothing domains for desired amount of softening effects. S-FEMs have now various forms, including cell-based smoothed finite element method (CS-FEM) [34], node-based smoothed finite element method (NS-FEM) [35], edge-based smoothed finite element method (ES-FEM) [36], and face-based smoothed finite element method (FS-FEM) [37]. Theoretical aspects of the S-FEMs were studied in [38–41]. In addition, investigation in [42,43] further discussed on the approximation of S-FEMs

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using non-mapped shape functions and its performance when elements are heavily distorted.

The numerical operations used in the S-FEMs bring in information from the neighboring elements in desired ways. Depending on the requirements of the analyst, the property of the S-FEM model can be distinct in various ways from that of the standard FEM model. The S-FEMs can also be viewed as a combination of the numerical treatments of both used in FEM and meshfree methods [29]. Since smoothing domains in the S-FEMs often cover part of the adjacent elements, the number of supporting nodes associated with the smoothing domain is larger than the number of nodes of an element. As a result, except the CS-FEM [33], the bandwidth of the stiffness matrix in the S-FEMs (NS-FEM, ES-FEM, SF-FEM) is increased, and the computational cost is hence higher than the standard FEM with the same degrees of freedom. On the other hand, thanks to the propagation of non-local information brought by the adjacent elements, the S-FEMs often produce much more accurate solutions than the standard FEM. For a given computational cost, all methods-FEM models gain better accuracy than that of the displacement-based FEM [1,2]. The S-FEM models were applied to a wide range of practical mechanics problems [44–58], and the S-FEM has become a simple and effective tool for analyzing a variety of practical problems. Among the existing S-FEM models, the ES-FEM was found so far the most computationally efficient [33].

The ES-FEM exhibits some interesting properties for solid mechanics problems such as: (1) it produces much more accurate solutions than the linear triangular elements (FEM-T3) and often found even more accurate than the FEM using quadrilateral elements (FEM-Q4) using the same sets of nodes; (2) the ES-FEM performs well with distorted meshes; (3) the ES-FEM is stable even for dynamic analysis and (4) it is quite simple to implement into the existing FEM packages without any additional degrees of freedom. However, the ES-FEM has also been found to have some inherent shortcomings for incompressible and nearly incompressible problems. Although the ES-FEM with help of the NS-FEM can solve volumetric locking as reported in [36], the ES/NS-FEM is a simple fix, but not a full proof in the incompressible limit. Our recent study on searching the root of the problem has revealed that the NS-FEM itself is not fully locking-free, as shown in this paper. Therefore, the performance of the ES-FEM for locking problems needs to be further investigated.

In this paper, we propose an improved formulation of the ES-FEM for analysis of solid mechanics problems. It is very simple to implement into the existing codes and has high effectiveness. The essential idea is to supplement the linear displacement fields with a so-called cubic bubble function associated with an internal node placed at the centroid of triangular elements. This injects some further softening into the bES-FEM model. The strain smoothing over the smoothing domains associated with edges of the elements is devised, and the smoothed Galerkin weak form is used to obtain the system stiffness matrix. We will verify that the weakened weak solution satisfies the divergence-free condition and hence works well for incompressible or nearly incompressible media. It is found that the present element is superior to other existing methods found in the literature, including the mixed displacement/pressure mixed model of the so-called MINI.

The rest of the paper is outlined as follows: Section 2 presents an improved formulation of the ES-FEM with a bubble function. Section 3 figures out an eigenvalue analysis of a number of methods in the incompressible limit. Displacement, energy and pressure error norms are defined in Section 4 for precise qualitative examination of various models. Several numerical examples are presented in Section 5. Section 6 concludes with some main remarks and discussions on directions for future work.

2. An improved formulation of the ES-FEM with bubble function

2.1. A brief on finite element formulation

Consider a two dimensional (2D) linear elastic solid defined in a domain Ω with a Lipschitz continuous boundary Γ. A body force $\mathbf{b}$ acts within the domain. Boundary Γ is split into two parts, namely $\Gamma_{\mathbf{u}}$ where displacements $\mathbf{u}$ are prescribed (Dirichlet conditions), and $\Gamma_{\mathbf{t}}$ where tractions $\mathbf{t}$ are prescribed (Neumann conditions). Those two parts form the boundary seamlessly $\Gamma = \Gamma_{\mathbf{u}} \cup \Gamma_{\mathbf{t}}$, $\Gamma_{\mathbf{u}} \cap \Gamma_{\mathbf{t}} = \varnothing$. The relations between the displacement field $\mathbf{u}$, the strain field $\boldsymbol{\varepsilon}$ and the stress field $\boldsymbol{\sigma}$ can be found in the literature [1,2].Let $\mathbb{V}$ and $\mathbb{V}_0$ be the two spaces of kinematically admissible displacements $\mathbf{u}$, respectively, defined by

$$\mathbb{V} = \{\mathbf{u} \in (\mathbb{H}^1(\Omega))^2,\ \mathbf{u} = \mathbf{u}_\Gamma \text{ on } \Gamma_D\} \tag{1}$$

$$\mathbb{V}_0 = \{\mathbf{v} \in \mathbb{V},\ \mathbf{v} = \mathbf{0} \text{ on } \Gamma_D\} \tag{2}$$

with a Hilbert space $\mathbb{H}^1(\Omega)$ defined in [3]. Let $\mathbb{V}^h \subset \mathbb{V}$ be a finite element approximation space. The statement of the discrete problem becomes finding a discrete solution $\mathbf{u}^h \in \mathbb{V}^h$ that satisfies [2]

$$\forall \mathbf{v}^h \in \mathbb{V}_0^h, \quad a(\mathbf{u}^h, \mathbf{v}^h) = f(\mathbf{v}^h) \tag{3}$$

where $a(.,.)$, $f(.)$ are the bilinear and linear forms, respectively, defined as

$$\begin{aligned} a(\mathbf{u},\mathbf{v}) &= 2\mu \int_\Omega \boldsymbol{\varepsilon}^T(\mathbf{u})\mathbf{D}_\mu \boldsymbol{\varepsilon}(\mathbf{v})\mathrm{d}\Omega + \lambda \int_\Omega \boldsymbol{\varepsilon}^T(u)\mathbf{D}_\lambda \boldsymbol{\varepsilon}(\mathbf{v})\mathrm{d}\Omega \\ &= \underbrace{2\ \mu \int_\Omega \boldsymbol{\varepsilon}^T(\mathbf{u})\mathbf{D}_\mu \boldsymbol{\varepsilon}(v)\mathrm{d}\Omega}_{\text{energy related to deviatoric term}} + \underbrace{\lambda \int_\Omega (\nabla \cdot \mathbf{u})(\nabla \cdot \mathbf{v})\mathrm{d}\Omega}_{\text{energy related to volumetric term}} \end{aligned} \tag{4}$$

$$f(\mathbf{v}) = \int_\Omega \mathbf{b}^T \mathbf{v}\mathrm{d}\Omega + \int_{\Gamma_t} \bar{\mathbf{t}}^{\mathbf{T}} \mathbf{v}\mathbf{d}\Gamma$$

with divergence $\nabla = \{\partial/\partial x, \partial/\partial y\}$, $\nabla \cdot (\bullet)$, and

$$\mathbf{D}_\mu = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0.5 \end{bmatrix}, \quad \mathbf{D}_\lambda = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \tag{5}$$

and $\mu = E/2(1+v),\ \lambda = vE/(1+v)(1-2v)$ are the Lamé constants.

It is well known that using lower-order finite elements leads to the poor performance in the incompressible limit. This is because the following constraint of divergence-free conditions is not satisfied:

$$\nabla \cdot \mathbf{u}^h \to 0 \text{ as } v \to 0.5 \tag{6}$$

This difficulty is well known to be the volumetric locking effect. This is due to the number of incompressible constraints is excessive to the total number of degrees of freedom in the finite element discretization, resulting in its corresponding ratio being less than one [3].

There are some ways to satisfy the divergence-free conditions. One of popular approaches is to introduce bubble functions [9,23] or residual-free bubbles [59,60] into mixed finite element approximations [3]. In our work, bubble functions are exploited and the method is simple to use only a displacement-based formulation.

2.2. A space for displacement field with bubble functions

Assume that the bounded domain Ω is discretized into a set $\mathfrak{T}$ of N_e elements and N_n nodes such that $\Omega \approx \Omega^h = \sum_{e=1}^{N_e} \Omega_e$. Let $\{N_I(\mathbf{x})\}$ be such nodal basis (functions) for the finite dimension space $\mathbb{V}^h$. As shown in [36], the shape functions used in the ES-FEM-T3 are