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Efficient finite element analysis using graph-theoretical force method; hexahedron elements

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ABSTRACT

Formation of a suitable null basis for equilibrium matrix is the main problem of finite elements analysis via force method. For an optimal analysis, the selected null basis matrices should be sparse and banded corresponding to sparse, banded and well-conditioned flexibility matrices. In this paper, an efficient method is developed for the formation of null bases of finite element models (FEMs) consisting of hexahedron elements, corresponding to highly sparse and banded flexibility matrices. This is achieved by associating special graphs with the FEM and selecting appropriate subgraphs and forming the self-equilibrating systems (SESs) on these subgraphs.

1. Introduction

The force method of structural analysis in which the member forces are used as unknowns is appealing to engineers since the properties of members of a structure most often depend on the member forces rather than joint displacements. This method was used extensively until 1960. The advent of the digital computer and the amenability of the displacement method for computation attracted most researchers. As a result, the force method and some of the advantages it offers in optimization and non-linear analysis and optimization has been neglected.

In the force method of analysis, the number of equations to be solved is the same as the degree of statical indeterminacy (DSI) of the model. While in the displacement method this number is the same as the degree of kinematical indeterminacy (DKI) which is also known the degrees of freedom (DOF). For some models with smaller DSI than DOF, it may be advantageous to utilize the force method. Immediate access to member forces which is required to be solved in reliability analysis is another advantage of the force method. For multiple redesign problems or nonlinear elastic analysis the force method allows the solution of the modified problems without restarting the computation from the beginning. For optimal design of a structure with fixed topology, it is often necessary to analyze the structure hundreds of times. In the force method since the statical basis stays unchanged for each design under different loading cases, it results in saving some computational time compared to the displacement approach. Apart from these positive points, any development of the force method as a dual approach to the displacement method can theoretically be attractive. For definition of the duality, the reader may refer to Argyris and Kelsey [1].

Five different approaches are adopted for the force method of structural analysis, classified as:

1. Topological force methods.
2. Graph theoretical methods.
3. Algebraic force methods.
4. Mixed algebraic-combinatorial force methods.
5. Integrated force method.

Topological methods have been developed by Henderson [2], Maunder [3] and Henderson and Maunder [4] for rigid-jointed skeletal structures. Graph theoretical methods based on cycle bases of the graph models are due to Kaveh [5,6]. These methods are generalized to cover different types of skeletal structures such as rigid-jointed frames, pin-jointed planar trusses and ball-jointed space trusses in [7,8].

Algebraic methods have been developed by Denke [9], Robinson [10], Topçu [11], Kaneko et al. [12], and Soyer and Topçu [13]. Mixed algebraic-topological methods have been used by Gilbert et al. [14] Coleman and Pothen [15,16], Pothen [17], and Heath et al. [18]. The integrated force method has been developed by Patnaik [19,20], in which the equilibrium and compatibility conditions

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are satisfied simultaneously in terms of the element force variables.

Recently applications of the graph theory methods are extended to two classes of finite element models. The first class takes the element forces along the edges of the elements [21–23] and in the second class the element forces are concentrated at the mid-edge of the edges of the elements [24].

In this paper, an efficient method is developed for the formation of null bases for finite element models comprising of hexahedron elements leading to highly sparse and banded flexibility matrices, and can be used for optimal finite element analysis by the force method. This is achieved by associating a special graph to the finite element model and selecting subgraphs (known as γ-cycles [7]) for the formation of localized self-equilibrating stress systems (null vectors). Their numerical values are calculated by an algebraic process. The efficiency and accuracy of the present method is illustrated through simple examples.

2. Formulation of force method

Consider a discrete or discretized structure which is statically indeterminate. The m-dimensional vector $\boldsymbol{r}$ contains independent element (member) forces, and an n-dimensional vector $\boldsymbol{p}$ denotes the nodal loads. The equilibrium equations of the structure can then be expressed as:

$$\mathbf{Ar} = \mathbf{p} \tag{1}$$

where $\boldsymbol{A}$ is an $n \times m$ equilibrium matrix. Assuming stability for the structure, the equilibrium matrix will have full rank, i.e. $t = m - n > 0$, $rank(A) = n$.

Also the member forces can be written as the sum of the particular and complementary solutions, where $\boldsymbol{q}$ is the t-dimensional vector of the redundant forces.

$$\mathbf{r} = \mathbf{B}_0\mathbf{p} + \mathbf{B}_1\mathbf{q} \tag{2}$$

$\boldsymbol{B}_0$ and $\boldsymbol{B}_1$ have m rows and n, and t columns, respectively. Pre-multiplying both sides of Eq. (2) by $\boldsymbol{A}$ and using Eq. (1) lead to

$$\mathbf{AB}_0 = \mathbf{I} \tag{3}$$

$$\mathbf{AB}_1 = \mathbf{0} \tag{4}$$

Here, $\boldsymbol{B}_0$ and $\boldsymbol{B}_1$ are not unique for a structure and many of such matrices can be formed. $\boldsymbol{B}_1$ is called static basis or self-stress matrix. This basis is known as null basis in mathematics and each column of the null basis matrix is known as a null vector. The null space and null vectors are mathematical counterparts of the complementary solution space and self-equilibrating systems, respectively.

Minimizing the complementary potential energy subjected to the constraint as in Eq. (1) requires $\boldsymbol{r}$ to minimize the quadratic form

$$\textbf{minimize}\left(\frac{1}{2}\mathbf{r}^t\mathbf{F_m}\mathbf{r}\right) \tag{5}$$

Here, $\boldsymbol{F_m}$ is a $m \times m$ block diagonal matrix known as the unassembled flexibility matrix containing the flexibility matrices of the elements of a structure in its block diagonal entries. Coupling Eq. (5) and Eq. (2) results in

$$\mathbf{q} = -\left(\mathbf{B}_1^t\mathbf{F_m}\mathbf{B}_1\right)^{-1}\left(\mathbf{B}_1^t\mathbf{F_m}\mathbf{B_0}\right)\mathbf{p} \tag{6}$$

According to Eq. (6) by solving a set of equations, redundant forces can be found.

After the determination of the member forces, using the load–displacement relationship for each member, one can write member distortion as

$$[\mathbf{u}] = [\mathbf{F_m}][\mathbf{r}] = [\mathbf{F_m}][\mathbf{B}_0\mathbf{B}_1]\begin{bmatrix}\mathbf{p}\\ \mathbf{q}\end{bmatrix} \tag{7}$$

Using virtual work, nodal displacements can be calculated as

$$[\mathbf{v}_0] = [\mathbf{B}_0^t][\mathbf{u}] \tag{8}$$

Combining Eqs. (7) and (8) leads to

$$\mathbf{v}_0 = \mathbf{B}_0^t\mathbf{F_m}\mathbf{B}_0\mathbf{p} + \mathbf{B}_0^t\mathbf{F_m}\mathbf{B}_1\mathbf{q} \tag{9}$$

Substituting Eq. (6) in Eq. (9) and using $\mathbf{D_{ij}} = \mathbf{B_i^t}\mathbf{F_m}\mathbf{B_j}$ leads to

$$\mathbf{v}_0 = [\mathbf{D}_{00} - \mathbf{D}_{01}\mathbf{D}_{11}^{-1}\mathbf{D}_{10}]\mathbf{p} = \mathbf{Fp} \tag{10}$$

Therefore the overall flexibility matrix of structure is obtained as

$$\mathbf{F} = \mathbf{D}_{00} - \mathbf{D}_{01}\mathbf{D}_{11}^{-1}\mathbf{D}_{10} \tag{11}$$

For free vibration of linear structure without damping we have

$$\left[[\mathbf{K}] - \boldsymbol{\omega}^2[\mathbf{M}]\right][\mathbf{v_0}] = \mathbf{0} \tag{12}$$

Obviously $\mathbf{Kv}_0 = \mathbf{p}$ and substituting Eq. (10) in Eq. (12) leads to

$$[[\mathbf{I}] - \boldsymbol{\omega}^2[\mathbf{m}][\mathbf{F}]][\mathbf{p}] = 0 \tag{13}$$

Then the frequency equation of the system in the force method is obtained as

$$[\mathbf{m}][\mathbf{F}] - \lambda[\mathbf{I}]| = 0 \text{ and } \lambda = 1/\boldsymbol{\omega}^2 \tag{14}$$

Efficiency of this analysis depends on the required time for the formation of the matrix $\mathbf{G} = \mathbf{B}_1^t\mathbf{F_m}\mathbf{B}_1$ and its characteristics, i.e. sparsity and bandedness together with its conditioning. For the formation of a well-structured matrix G, one should select a well-structured $\boldsymbol{B}_1$ matrix.

Many algebraic procedures based on various matrix factorizations such as Gauss-Jordan elimination, LU, QR, LQ exist for the formation a null basis matrix $\boldsymbol{B}_1$ of an equilibrium matrix $\boldsymbol{A}$ [14,18,27]. Basic concept of these methods is described briefly in the following. Let matrix $\boldsymbol{A}$ be partitioned using a column permutation matrix $\boldsymbol{P}$ as below:

$$\mathbf{AP} = [\mathbf{A}_1, \mathbf{A}_2] \tag{15}$$

where $\boldsymbol{A}_1$ is a $n \times n$ non-singular matrix. Obviously matrix $\boldsymbol{B}_1$ can be written as

$$\mathbf{B}_1 = \mathbf{P}\begin{bmatrix}-\mathbf{A}_1^{-1}\mathbf{A}_2\\ \mathbf{I}\end{bmatrix} \tag{16}$$

2.1. LU decomposition method

Using the LU decomposition method, one obtains the LU factorization of **A** as::

$$\mathbf{PA} = \mathbf{LU} \text{ and } \mathbf{U}\bar{\mathbf{P}} = [\mathbf{U}_1, \mathbf{U}_2] \tag{17}$$

P and $\bar{\mathbf{P}}$ are permutation matrices of order $n \times n$ and $m \times m$, respectively. Now $\mathbf{B}_0$ and **B** can be written as:

$$\mathbf{B}_0 = \bar{\mathbf{P}}\begin{bmatrix}-\mathbf{U}_1^{-1}\mathbf{L}^{-1}\mathbf{p}\\ 0\end{bmatrix} \quad \text{and} \quad \mathbf{B}_1 = \bar{\mathbf{P}}\begin{bmatrix}-\mathbf{U}_1^{-1}\mathbf{U}_2\\ \mathbf{I}\end{bmatrix} \tag{18}$$

2.2. QR decomposition method

Using a **QR** factorization algorithm with column pivoting yields,

where **P** is again a permutation matrix, and R_1 is an upper triangular matrix of order n. B_1 can be obtained as:

$$\mathbf{AP} = \mathbf{Q}[\mathbf{R}_1, \mathbf{R}_2] \tag{19}$$

$$\mathbf{B}_1 = \mathbf{P}\begin{bmatrix}-\mathbf{R}_1^{-1}\mathbf{R}_2\\ \mathbf{I}\end{bmatrix} \tag{20}$$