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The value of numerical amplification matrices in time integration methods

José María Benítez *, Francisco Javier Montáns

Escuela Técnica Superior de Ingenieros Aeronáuticos, Universidad Politécnica de Madrid, Pza.Cardenal Cisneros, 3, 28040 Madrid, Spain

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ABSTRACT

Several frequently overlooked concepts in linear elastodynamics are reviewed. First that the amplification matrix may be obtained numerically and if obtained this way, it may give some extra information on the programmed algorithm. Second that the adequate dimension of that matrix depends on the algorithm at hand. Such dimension equals the number of independent initial conditions that must be prescribed. Third that those initial conditions should be consistent with the problem at hand *and* with the algorithm. Overshooting phenomena present in some time integration algorithms may be a consequence of overlooking such issues.

1. Introduction

In linear and nonlinear transient dynamics finite element analysis, time integration algorithms need to be employed. In explicit analysis the central difference method is the common choice. In implicit formulations, the Newmark-β method and specifically the trapezoidal rule, is still the most common selection. There are of course many other alternatives. Some of them are quite old and well known and may be found in general purpose finite element books, see for example [1–3]. Some are being currently proposed [6–9]. Among the traditional ones is the Wilson-θ method [10]. This method, which originally uses the linear acceleration formulae, can be generalized using the Newmark-β equations to yield a more general Wilson collocation method [11] of which the Wilson-θ method is a particular case. This algorithm shows some desirable features as the numerical damping of high frequency modes. However, the Wilson-θ method, which is considered self-starting [1], [5] has been seldom used because it is known to present *"overshooting"* during the first steps when using a large time increment relative to the period of the problem (or mode) if some initial conditions are used [2,12–14]. This is a highly undesirable feature because the method seems unreliable for those high frequency modes we want to numerically damp. A similar although mild overshooting has also been encountered in the Hilber–Hughes–Taylor (HHT) method [2], also considered self-starting [15]. Stability of the time integration algorithms in linear dynamics is usually studied through the amplification matrix of a single degree of freedom problem because modal decomposition and proportional damping converts a $n-$ degree of freedom system in n uncoupled single degree of freedom systems. In the case of Wilson methods, the spectral radius of the amplification matrix predicts stable and damped response for those high frequency modes if the θ collocation parameter is correctly chosen. A possible explanation for the reason of the overshooting that Wilson-θ and HHT methods present is given in Reference [2]. However, we will show that a more detailed analysis of overshooting phenomena can be performed using numerically obtained amplification matrices. In fact, it will be shown that numerically obtained amplification matrices give in general more accurate information about the programmed algorithm in the short (few steps) and long term (many steps) and also about the consistency of the initial conditions. In some sense numerically derived amplification matrices can be considered to be more representative than analytical ones because they exactly mimic what the programmed algorithm does in non-exact arithmetic. We have chosen the well known Wilson-θ algorithm to explain and illustrate these issues. We also show that the same concepts are applicable to the HHT algorithm, but many conclusions may be applicable to other algorithms as well.

2. The Collocation Wilson method

In this section we briefly summarize the Collocation Wilson-Newmark formulae to introduce notation and for future reference. Let $\boldsymbol{K}$,$\boldsymbol{M}$, $\boldsymbol{C}$ be the stiffness, mass and damping matrices and let $\boldsymbol{f}$

* Corresponding author. Tel.: +34 913366367; fax: +34 91366366.
E-mail addresses: josemaria.benitez@upm.es (J.M. Benítez), fco.montans@upm.es (F.J. Montáns).

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and $\boldsymbol{u}$ be the load and displacement vectors respectively. Equilibrium is established at $n+\theta$ with $\theta > 0$ [1,2,11]

$$\boldsymbol{M}\ddot{\boldsymbol{u}}_{n+\theta} + \boldsymbol{C}\dot{\boldsymbol{u}}_{n+\theta} + \boldsymbol{K}\boldsymbol{u}_{n+\theta} = \boldsymbol{f}_{n+\theta} \tag{1}$$

where θ is a parameter that is usually $\theta \geqslant 1$, and the acceleration and external loads $\ddot{\boldsymbol{u}}_{n+\theta}$, $\boldsymbol{f}_{n+\theta}$ at $n+\theta$ are defined by the following relation those at steps n and $n+1$.

$$\ddot{\boldsymbol{u}}_{n+\theta} = (1-\theta)\ddot{\boldsymbol{u}}_n + \theta\ddot{\boldsymbol{u}}_{n+1} \tag{2}$$

$$\boldsymbol{f}_{n+\theta} = (1-\theta)\boldsymbol{f}_n + \theta\boldsymbol{f}_{n+1} \tag{3}$$

Then, to obtain $\dot{\boldsymbol{u}}_{n+\theta}$ and $\boldsymbol{u}_{n+\theta}$, the Newmark-$\beta$ integration formulae are used extended to $n+\theta$, i.e. the time increment is $\theta\Delta t$

$$\dot{\boldsymbol{u}}_{n+\theta} = \dot{\boldsymbol{u}}_n + \theta\Delta t[(1-\gamma)\ddot{\boldsymbol{u}}_n + \gamma\ddot{\boldsymbol{u}}_{n+\theta}] \tag{4}$$

$$\boldsymbol{u}_{n+\theta} = \boldsymbol{u}_n + \theta\Delta t\dot{\boldsymbol{u}}_n + \frac{1}{2}(\theta\Delta t)^2[(1-2\beta)\ddot{\boldsymbol{u}}_n + 2\beta\ddot{\boldsymbol{u}}_{n+\theta}] \tag{5}$$

where β and γ are the Newmark integration parameters.

Substituting Eq. (2) in Eqs. (4) and (5), we can write in predictor–corrector format

$$\dot{\boldsymbol{u}}_{n+\theta} = \underbrace{\dot{\boldsymbol{u}}_n + \theta\Delta t(1-\theta\gamma)\ddot{\boldsymbol{u}}_n}_{\text{predictor:}\dot{\boldsymbol{u}}^p_{n+\theta}} + \underbrace{\gamma\theta^2\Delta t\ddot{\boldsymbol{u}}_{n+1}}_{\text{corr.:}\dot{\boldsymbol{u}}^c_{n+\theta}} \tag{6}$$

and

$$\boldsymbol{u}_{n+\theta} = \underbrace{\boldsymbol{u}_n + \theta\Delta t\dot{\boldsymbol{u}}_n + \frac{1}{2}(\theta\Delta t)^2(1-2\theta\beta)\ddot{\boldsymbol{u}}_n}_{\text{predictor:}\boldsymbol{u}^p_{n+\theta}} + \underbrace{\beta\theta^3\Delta t^2\ddot{\boldsymbol{u}}_{n+1}}_{\text{corr.:}\boldsymbol{u}^c_{n+\theta}} \tag{7}$$

These two equations may be substituted into Eq. (1) to obtain the following equation in *a-form*

$$\boldsymbol{M}^*\ddot{\boldsymbol{u}}_{n+1} = \boldsymbol{f}^*_{n+1} \tag{8}$$

where

$$\begin{aligned}\boldsymbol{M}^* &= \theta\boldsymbol{M} + \gamma\theta^2\Delta t\boldsymbol{C} + \beta\theta^3\Delta t^2\boldsymbol{K}\\ \boldsymbol{f}^*_{n+1} &= \boldsymbol{f}_{n+\theta} - (1-\theta)\boldsymbol{M}\ddot{\boldsymbol{u}}_n - \boldsymbol{C}\dot{\boldsymbol{u}}^p_{n+\theta} - \boldsymbol{K}\boldsymbol{u}^p_{n+\theta}\end{aligned} \tag{9}$$

Once $\ddot{\boldsymbol{u}}_{n+1}$ is solved for, the velocities $\dot{\boldsymbol{u}}_{n+1}$ and the displacements $\boldsymbol{u}_{n+1}$ at $n+1$ are obtained from Newmark's formulae with Δt as time increment

$$\dot{\boldsymbol{u}}_{n+1} = \dot{\boldsymbol{u}}_n + (1-\gamma)\Delta t\ddot{\boldsymbol{u}}_n + \gamma\Delta t\ddot{\boldsymbol{u}}_{n+1} \tag{10}$$

$$\boldsymbol{u}_{n+1} = \underbrace{\boldsymbol{u}_n + \Delta t\dot{\boldsymbol{u}}_n + \frac{\Delta t^2}{2}(1-2\beta)\ddot{\boldsymbol{u}}_n}_{\boldsymbol{u}^p_{n+1}} + \beta\Delta t^2\ddot{\boldsymbol{u}}_{n+1} \tag{11}$$

The original Wilson θ method employs the linear acceleration constants $\beta = 1/6$ and $\gamma = 1/2$.

3. Numerical amplification matrix

As it is well known, see for example [1,2], linear multistep methods may be written in the form

$$\boldsymbol{y}_{n+1} = \boldsymbol{A}\boldsymbol{y}_n + \boldsymbol{L}_n \tag{12}$$

where $\boldsymbol{L}_n$ is the equivalent algorithmic load vector and $\boldsymbol{A}$ is the algorithmic amplification matrix. In the absence of external loads, the algorithm is governed by $\boldsymbol{y}_{n+1} = \boldsymbol{A}\boldsymbol{y}_n = \boldsymbol{A}^n\boldsymbol{y}_1$ and, hence, the amplification matrix governs the stability of the numerical predictions [1]. The vector $\boldsymbol{y}_n$ depends on the specific algorithm and may be properly customized. A typical choice is the usual dynamics state vector

$$\boldsymbol{y}_n = [\boldsymbol{u}_n, \boldsymbol{v}_n]^T \tag{13}$$

where $\boldsymbol{u}_n$ and $\boldsymbol{v}_n$ are the displacements and velocities respectively. The linear case is usually selected to study the properties of the integration algorithms. In such case modal superposition applies and the stability properties are given by those of the mode with highest frequency ω (or equivalently lowest period T). Hence, the amplification matrix is usually obtained analytically for one single degree of freedom and plots for the spectral radii $\rho(\boldsymbol{A})$ are obtained as those given in Fig. 1 for different integration methods: Wilson-θ, collocation Wilson-θ schemes, trapezoidal rule, Hilber–Hughes–Taylor (HHT) method and Bathe method. However, the analytical derivation of the amplification matrix is a tedious task, so sometimes the amplification matrices for a method are published after the method has been introduced and some properties of the method analyzed. For example, for the recently published Bathe algorithm, the spectral radius clearly shows the previously observed numerical superiority of the algorithm in terms of stability respect to other algorithms because the $\Delta t/T$ values for spectral radius decay is larger than in other algorithms, preserving important information at those frequencies, and for $\Delta t/T \to \infty$ then $\rho(\boldsymbol{A}) \to 0$, completely damping the response of the highest modes in one step. These are frequently desirable features, which in this case add to the implementation simplicity of the method. A similar explicit algorithm which also shows better properties than the central differences method may be found in [9].

In this paper we advocate the numerical determination of the amplification matrix. This matrix may be readily determined applying the algorithm to two single degree of freedom problems with zero loads and with initial conditions $\boldsymbol{y}_0$ and $\bar{\boldsymbol{y}}_0$ respectively, defined from the initial displacement u_0 and initial velocity v_0 as

$$\boldsymbol{y}_0 = \begin{bmatrix} u_0 \\ v_0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \text{ and } \bar{\boldsymbol{y}}_0 = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \tag{14}$$

so the response obtained in just one step performed using the time integration algorithm at hand is by definition of $\boldsymbol{A}$

$$\boldsymbol{y}_1 = \boldsymbol{A}\boldsymbol{y}_0 = \begin{bmatrix} A_{11} & A_{21} \\ A_{12} & A_{22} \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} A_{11} \\ A_{12} \end{bmatrix} \tag{15}$$

$$\bar{\boldsymbol{y}}_1 = \boldsymbol{A}\bar{\boldsymbol{y}}_0 = \begin{bmatrix} A_{11} & A_{21} \\ A_{12} & A_{22} \end{bmatrix}\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} A_{21} \\ A_{22} \end{bmatrix} \tag{16}$$

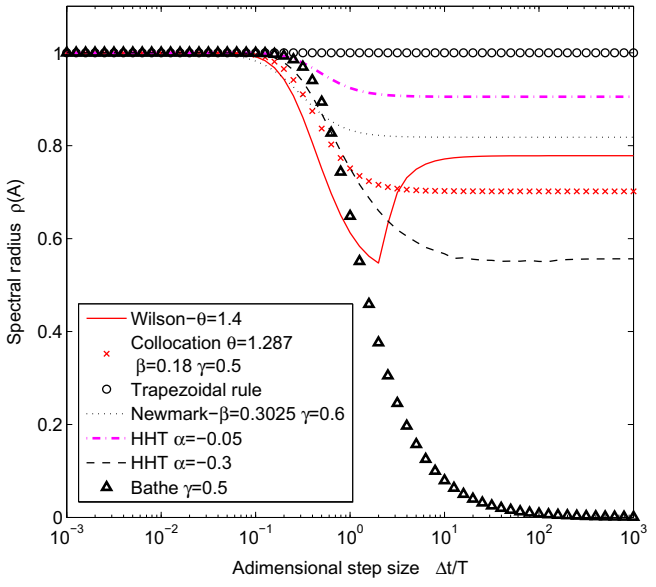

Fig. 1. Spectral radii for different methods.