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Confidence extremal structural response analysis of truss structures under static load uncertainty via SDP relaxation

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ABSTRACT

A new method is presented to obtain the confidence structural responses of truss structures under ellipsoid static load uncertainty. By using a combination of duality theorem and SDP relaxation technique, we reformulate the original convex maximization problem as a relaxed convex SDP problem, which can be solved with global optimality. Numerical examples demonstrate the effectiveness of the proposed approach.

1. Introduction

In structural and mechanical design, traditional optimization methods based on deterministic model have been applied successfully. In deterministic optimization model, parameters such as material properties, loads and geometry coordinates are all treated as deterministic parameters with fixed values. However, uncertainties such as manufacturing and observation errors do exist when a structure is built. Therefore the concept of uncertainty must be incorporated into the model of optimization otherwise the reliability of the optimal design can not be guaranteed.

A lot of robustness-based optimization methods have been proposed in last two decades. Generally speaking these methods are mainly based on two kinds of uncertainty models – the probabilistic uncertainty model [1–5] and the non-probabilistic uncertainty model [6–9]. Using the non-probabilistic model, generally, the robust optimization problem can be formulated as follows:

$$\begin{aligned} \min_{\boldsymbol{x}} \quad & \max_{\mathbf{p}\in U_p} f(\boldsymbol{x};\mathbf{p}) \\ \text{s.t.} \quad & \max_{\mathbf{p}\in U_p} g(\boldsymbol{x};\mathbf{p}) \leqslant 0, \quad i = 1,\ldots,n, \end{aligned} \tag{1.1}$$

where $\boldsymbol{x}$ and $\mathbf{p}$ denote the design variable vector and uncertain parameter vector, respectively. U_p is a bounded set in the space of $\mathbf{p}$. Eq. (1.1) represents a nested Bi-level optimization problem. At the upper level, the aim is to find the best design to improve the structural performance, while at the lower level the aim is to find the worst structural responses (extremal structural responses) considering the uncertainty of the parameters for a given design. It means that the feasibility of a given design should be determined by solving another optimization problem at this stage. This is quite different from deterministic optimization problems. It is worth noting that the requirements of the global optimality of the solutions for the upper and lower level optimization are not the same. For the upper level optimization, a local optimum is acceptable since improvements can be still made even without global optimality. On the other hand, for the lower level problem, the global optimality of the solution *must be satisfied* otherwise the feasibility of a given design can not be guaranteed, since *worst case response* (*extremal responses*) must be considered in this case. In other words, in order to solve the robust optimization problem formulated in the form of Bi-level program correctly, the global optimality of the the lower level problem is a necessary requirement. Unfortunately, this issue has not been well addressed in the literatures.

As for finding the extremal values of the structural response under the uncertainty of parameters, various of approaches have been proposed. Assuming the uncertain parameters perturbing within small intervals, Qiu and Elishakoff [10] made an interval analysis of structures based on the first-order interval perturbation approach. Muhanna and Mullen [11] made further discussions on the linear interval approach for static structure analysis with uncertain parameters. Rao and Berke [12] developed an approach for the analysis of engineering systems for which the input parameters are given as interval members. Mcwilliam [13] discussed how to calculate the static displacement bounds of structures with uncertain parameters. Based on the perturbation technique and

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the interval extension, Chen et al. [14] proposed an approach to find the upper and lower bounds of the static displacements. Recently, with the use of quadratic embedding of uncertainty and the semi-definite relaxation techniques, Kanno et al. [15,16] proposed an elegant approach to compute the robustness function and confidence ellipsoids for static response of truss structures with load and structural uncertainties. For the tested problems, it was reported that tight bounds can be obtained even for moderately large magnitudes of perturbations.

The aim of the present paper is to discuss how to guarantee the feasibility of a given design transferred from the upper level problem by solving the lower level *worst case analysis* optimization problem in robust optimization of truss structures under static load uncertainty. As will be shown in Section 3, for the considered case, the lower level problem is in fact a *concave minimization* (or *convex maximization*) program, which has been studied intensively in global optimization. Generally speaking, *concave minimization* problems are NP-hard and will possess many local optimal solutions. For this reason, they are also called as multi-extremal global optimization problems [17]. Using standard algorithms, it will fail to find the global optimal solution of the concave minimization problems, which has been pointed out by many researchers.

In order to overcome the difficulties mentioned above, in this paper we reformulate the problem as a convex SDP problem, with the use of the SDP relaxation technique. Our idea is to find a confidence upper bound of the global optimal solution in the lower level problem instead of trying to find the global optimal solution directly. In this way, the feasibility of a given design can always be assured. The obtained SDP problem is a convex program, which can be solved with global optimality by well established algorithms. The rest of this paper is organized as follows: In Section 2, the formulation of the considered robust optimization problem is proposed with its properties being discussed. In Section 3, we show that the non-convex lower level optimization problem can be reformulated as a standard linear SDP problem. This SDP problem is exactly the dual of the primal optimization problem. It can be shown that if the uncertainty of the external load is described by a single ellipsoid, then there is no dual gap between the two problems although the primal problem is non-convex i.e., the optimal solutions of the primal problem can be obtained accurately by solving the dual problem instead. If the uncertainties are described by multiple ellipsoids, then confidence estimations of the extremal structural responses can be obtained, which is very important to ensure the feasibility of a given design point. The approach proposed is then applied to several test problems for demonstration of its effectiveness in Section 4. Finally some concluding remarks are given.

2. Problem formulation

2.1. Notations

Let $\mathbb{S}^n \subset \mathbb{R}^{n\times n}$ be the space of real symmetric matrices. A matrix $\mathbf{X} \in \mathbb{S}^n$ is said to be positive semi-definite or positive definite if $\boldsymbol{x}^\top \mathbf{X}\boldsymbol{x} \geqslant 0$ or $\boldsymbol{x}^\top \mathbf{X}\boldsymbol{x} > 0\ \forall \boldsymbol{x} \in \mathbb{R}^n$. Equivalently, $\mathbf{X} \in \mathbb{S}^n$ is positive semi-definite if all its eigenvalues are nonnegative. $\mathbf{X} \succ 0$ ($\mathbf{X} \succeq 0$) means $\mathbf{X}$ is positive definite (positive semi-definite). These notations will be used in the subsequent texts.

2.2. Robust optimization of truss structure under load uncertainty

It is well known that the structural robust optimization problem can be described as a Bi-level program. If the minimization of the total weight of a truss structure is considered with the uncertain external load, the optimization problem can be written as

$$\text{find} \quad \boldsymbol{a} := (a_1, \ldots, a_n)^\top \in \mathbb{R}^n$$

$$\text{Min} \quad w = \sum_{i=1}^{n} \varrho_i l_i a_i \tag{2.1a}$$

$$\text{s.t.} \quad \max_{\mathbf{p}\in U_p} g_i(\boldsymbol{a};\mathbf{p}) \leqslant 0, \quad i = 1,\ldots,m, \tag{2.1b}$$

where n is the total number of bars in the ground structure, m denotes the number of the behavior constraints. ϱ_i and l_i are the mass density and the length of the ith bar, respectively. $\boldsymbol{a} := (a_1,\ldots,a_n)^\top \in \mathbb{R}^n$ is the vector of design variables with a_i denoting the cross sectional area of the ith bar. $\mathbf{p} \in \mathbb{R}^{n^d}$ is the external load vector with n^d denoting the total number of the degrees of freedom. We first assume that the uncertainty of the external load can be described by a single ellipsoid:

$$U_p = \{\mathbf{p} \in \mathbb{R}^{n^d} | (\mathbf{p}-\mathbf{p}_0)^\top \mathbf{B}(\mathbf{p}-\mathbf{p}_0) \leqslant 1\}, \tag{2.2}$$

where $\mathbf{B} \in \mathbb{R}^{n^d\times n^d}$ is a symmetric positive semi-definite matrix and $\mathbf{p}_0 \in \mathbb{R}^{n^d}$ is the nominal value vector of the external load. In the next section, we will discuss the more general case in which the uncertainties are described by multiple-ellipsoids.

The lower level optimization problem appeared in (2.1b) determines the feasibility of a given design point $\boldsymbol{a}$. This is achieved by finding the extremal (worst case) structural response as $\mathbf{p}$ is varied in U_p. As has been pointed out in the previous section, it is of utmost importance to solve (2.1b) with global optimality. In the following, we will discuss the properties of the lower level problem (2.1b) from the global optimization point of view. It can be shown that for different kinds of structural responses, the degree of difficulty for the corresponding worse case analyses will be quite different.

If the considered behavior constraint in (2.1b) is the algebraic value of a nodal displacement, then (2.1b) can be expressed as(assuming that the global optimal value is greater than zero):

$$\begin{aligned} P_1 \quad & \text{find} \quad \mathbf{p} \in \mathbb{R}^{n^d} \\ & \text{Min} \quad -\mathbf{d}_l^\top \mathbf{K}^{-1}\mathbf{p} \\ & \text{s.t.} \quad (\mathbf{p}-\mathbf{p}_0)^\top \mathbf{B}(\mathbf{p}-\mathbf{p}_0) \leqslant 1, \end{aligned} \tag{2.3}$$

where $\mathbf{K} \in \mathbb{S}^{n^d}$ is the global stiffness matrix. $\mathbf{d}_l \in \mathbb{R}^{n^d}$ is the localization vector such that $\mathbf{d}_l^\top \mathbf{u} = u_l$ with u_l denoting the lth component of the displacement vector.

The objective function in P_1 is linear with respect to $\mathbf{p} \in \mathbb{R}^{n^d}$ and the feasible domain of P_1 is a convex set. Therefore P_1 is a convex optimization problem, which can be solved by general gradient based optimization algorithms with global optimality.

Using the algebraic value of a nodal displacement as objective function in P_1 implies that the direction of the concerned nodal displacement is known a prior. This is only reasonable when the magnitude of the disturbance of the external load around its nominal value is not very large. If large magnitude of uncertainty is considered, it is more suitable to use the Euclidean norm of a displacement vector as the measure of the extremal structural response. Under this circumstance, the lower level problem can be formulated as

$$\begin{aligned} P_2 \quad & \text{find} \quad \mathbf{p} \in \mathbb{R}^{n^d} \\ & \text{Min} \quad -(u_l^2 + u_m^2) = -\mathbf{p}^\top \mathbf{A}\mathbf{p} = -\mathbf{p}^\top(\mathbf{A}_l + \mathbf{A}_m)\mathbf{p} \\ & \text{s.t.} \quad (\mathbf{p}-\mathbf{p}_0)^\top \mathbf{B}(\mathbf{p}-\mathbf{p}_0) - 1 \leqslant 0, \end{aligned} \tag{2.4}$$

where $\mathbf{A}_l = \mathbf{K}^{-1}\mathbf{d}_l\mathbf{d}_l^\top\mathbf{K}^{-1} \succeq 0$ and $\mathbf{A}_m = \mathbf{K}^{-1}\mathbf{d}_m\mathbf{d}_m^\top\mathbf{K}^{-1} \succeq 0$.

Similarly, if the compliance of the structure is considered, the corresponding optimization problem is