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Measuring unfairness feeling in allocation problems ☆

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ABSTRACT

In this paper, we introduce a new measure of social fairness based on unfairness feelings of the players involved in an allocation problem, e.g., cake-cutting problem or shift scheduling. We only require that each player be described by a von Neumann–Morgenstern utility function. Next, we propose a social normalization of each player's utility function, based on how each player sees the other players' shares through her own utility function. Further, we extend this normalization idea to a setting where the players are represented by a weighted oriented graph, where the weights assess the relatedness of (or similarities between) the agents. Among other results, we establish some links between our measures of fairness and those classically used in the cake-cutting-problem literature.

1. Introduction

The problem of the fair division of a cake, a metaphor used to designate a common resource, has been the topic of a large body of literature in the last six decades or so; see, e.g., Steinhaus [20], Brams and Taylor [4], Brams and Taylor [5], Robertson and Webb [19]. Typically in this literature, two assumptions have been made on the individual utility functions of the stakeholders in the cake, namely: (i) they are additive; and (ii) the utility of the whole cake is (normalized to) one for all players. These assumptions have been instrumental in designing cake-cutting algorithms and deriving some properties. Further, solving such problems requires us to specify from the outset what is meant by a *fair* division. Here, the literature has proposed a series of definitions of fairness, e.g., exact, proportional, envy-free and equitable fairness, each having its pros and cons.[1]

In this paper, we introduce a new measure of fairness without requiring that the utility functions be additive or that the whole-cake value be normalized to one for all players. The original motivation for developing this measure was a research contract to provide a methodology for shift-scheduling problems where a manager wishes to implement the (technically feasible) schedule that minimizes a certain unfairness criterion. The starting assumption is that the manager can obtain the employees' preferences regarding a small[2] set of acceptable schedules.[3] In such a context, the additivity assumption naturally does not hold anymore, that is, the utility of the sum of two shifts is clearly not equal to the sum of their utilities, and the whole-cake-normalization assumption is meaningless. The idea that the utility function is not necessarily additive, but rather super- or sub-additive, is by no means specific to shift scheduling but is a standard assumption in economics. The implications of abandoning the additivity assumption are important. In particular, Mirchandani [13] showed that most existing fair-division procedures are incompatible with nonadditive utility functions.

In this paper, we do not require additivity of utility functions and only assume that each player has a von Neumann–Morgenstern (vNM) utility function. To be able to compare players' payoffs and adequately assess the fairness of any division of the cake, we propose a normalization of the individual utility functions. As we will see, this normalization is centered on the idea that each player compares its allocation to other players' allocations through the lens of its own utility function. For this reason, we call it a social normalization. Next, using socially normalized utility functions, we introduce the concept of the *unfairness feeling*. A division will then be called socially fair if all players have no unfairness feeling.

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[1] Of course, many other measures of fairness and equity exist and are based on different premises. For an interesting discussion in the context of resource allocation, the interested reader may refer to Bertsimas et al. [3].

[2] Pragmatically, it does not make sense to ask the employees to rate all feasible schedules.

[3] By acceptable, we mean a technically feasible schedule that does not involve a too high additional cost with respect to the least-cost one. Put differently, management is willing to forgo some revenues in order to please the employees.

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We will relate our social fairness to classical cake-cutting-literature definitions. In problems with a large number of agents, or when the agents are heterogenous in some way, it may become intuitively appealing to suppose that each stakeholder in the cake is only sensitive to how "similarly" or "closely" players are treated. To handle such a case, we extend our definition of fairness to a setting where the players are located on a weighted oriented graph. This formulation captures the idea of a social network where individuals are only interested in the fate of those in their circles.

The rest of the paper is organized as follows. Section 2 provides some background and preliminaries. Sections 3 and 4 introduce our normalization of utilities and concepts of fairness, respectively. Section 5 deals with local fairness, and Section 6 briefly concludes.

2. Background and preliminaries

We start by recalling some of the most commonly used definitions of fairness in the cake-cutting-problem literature, to which we will link our fairness criterion:

Exact fairness: A division is exact if all players' allocations are identical, i.e., exchanging shares will not affect any player's outcome.

Proportionality (*Pro*): A division is proportionally fair[4] if every player prefers its allocation to an allocation from an exact division. If we suppose that the cake is fully allocated among n agents, then Pro can be interpreted as an allocation where each agent prefers its share to the average of what they would get if allocations were given away uniformly randomly. See, e.g., Procaccia [18], Mossel and Tamuz [14], Bei et al. [2].

Envy-freeness (*EF*): A division is envy-free if every player prefers its allocation to any other player's allocation. See the early contributions in, e.g., Foley [10], Varian [23], Varian [24], Arnsperger [1], and the more recent ones focusing on cake-cutting procedures in, e.g., Stromquist [21], Cohler et al. [8], Chen et al. [7].

Equitable fairness: A division is equitable if all players have the same utility for their respective shares.

To illustrate some of the drawbacks in these definitions, we consider a few anecdotal examples. When the cake is made of indivisible pieces, e.g., where a car and a summer cottage to be fairly shared following a divorce, an exact division is obviously not implementable. Even when it is, exactness can be very restrictive and lead to some counter-intuitive results. For instance, if a cake that contains chocolate on a half and nuts on the other half is to be shared between a person allergic to nuts and another who hates chocolate, then imposing an exact division would be peculiar and obviously not Pareto-optimal (assuming that this feature is of interest). Proportional fairness and envy-free fairness are much less demanding than exactness, but may also be infeasible when the goods are indivisible. Equitable fairness may be questionable on some grounds. To see this, consider a sugar cake with three cherries on top, to be divided among four individuals, three of whom have no interest at all in the cake but love cherries, while the fourth person only wants the sugar cake. One division is to give to the first three players one cherry, and the cake to the fourth person. This solution is intuitively fair, and it is fair according to the definitions of Pro and EF, but it is not equitable. This highlights that equitable fairness may fail to achieve a fair solution according to common sense. In these examples, the focus was on a "physical" division of the cake rather than on dividing the corresponding total value of the cake. For this, we must define a utility function for each player that has certain properties. This is where the assumptions mentioned in the introduction come into play, namely, the additivity and normalization to one of the whole cake.

It is noteworthy that important variants of this setting have been widely studied in the cake-cutting literature. On one hand, Gardner [11] introduced the chore division problem. In this problem, the cake stands for chores that agents would want not to have, but must nevertheless be fully allocated. It is not too hard to see that the definitions stated above still apply straightforwardly to this setting. Interestingly, Su [22], Peterson and Su [16] and Peterson and Su [17] have shown how cake-cutting methods could be adapted to chore division. On the other hand, recently, models by Brânzei et al. [6], Li et al. [12] and Velez [25] have allowed for an agent's utility to also depend on other agents' allocations. This externality typically captures the fact that an agent may care about another agent's fate, and it can be positive or negative. We now introduce the notation and formally discuss these issues.

In a cake-cutting problem, a finite set of players $N=\{1,\dots,n\}$ and a cake CAKE are given.[5] A division, or allocation, of the cake is a vector $x=(x_1,\dots,x_n)$, where $x_i \subset \text{CAKE}$ is the share of player i, and $\bigcup_{i\in N} x_i = \text{CAKE}$. Each player i has a utility function u_i that associates a real number to any x_i. Player i prefers share x_i to x_i', if, and only if, $u_i(x_i) \geq u_i(x_i')$. The utility function is additive; that is, for any disjoint subsets x_k and x_l, we have

$$u_i(x_k \cup x_l) = u_i(x_k) + u_i(x_l). \quad (1)$$

In particular, this implies that the utility of an empty allocation is equal to zero, i.e., $u_i(\varnothing)=0$, and by the normalization of the whole-cake value, we have $u_i(\text{CAKE})=1$, $\forall i \in N$. With this notation, we can rephrase the above definitions of fairness as follows:

Exact division: A division is exact if for any player $i \in N$, $u_i(x_j) = 1/n$, $\forall j \in N$.

Proportionality: A division is proportionally fair (Pro) if any player gets at least $1/n$, i.e., $u_i(x_i) \geq 1/n$, $\forall i \in N$.

Envy-Freeness: A division is envy-free (EF) if any player i prefers its allocation to any other player's allocation, i.e., $u_i(x_i) \geq u_i(x_j)$, $\forall\, j \in N$.

Equitable: A division is equitable if all players obtain the same utility, i.e., $u_i(x_i) = u_j(x_j)$, $\forall i,j \in N$.

Suppose now that a given set of feasible divisions of the cake are proposed to the agents and they are asked to rate them according to their utility functions. For any allocation $x=(x_1,\dots,x_n)$, the resulting evaluation can be represented by the following utility matrix:

$$U = \begin{pmatrix} u_1(x_1) & \dots & \dots & \dots & u_1(x_n) \\ \vdots & \ddots & & & \vdots \\ \vdots & & u_i(x_i) & & \vdots \\ \vdots & & & \ddots & \vdots \\ u_n(x_1) & \dots & \dots & \dots & u_n(x_n) \end{pmatrix} = \begin{pmatrix} U_{11} & \dots & \dots & \dots & U_{1n} \\ \vdots & \ddots & & & \vdots \\ \vdots & & U_{ii} & & \vdots \\ \vdots & & & \ddots & \vdots \\ U_{n1} & \dots & \dots & \dots & U_{nn} \end{pmatrix}, \quad (2)$$

where $(u_i(x_j))_{1 \leq i,j \leq n}$ (or U_{ij}) gives the utility of player i for player j' s share. An interesting feature of the utility matrix is that it

[4] This concept is not to be confused with proportional fairness that appears, for instance, in Nash Jr [15], Bertsimas et al. [3] and Cole et al. [9], which, roughly, corresponds to maximizing the (weighted) product of the agents' utilities. Note that, for this concept to be well-defined, a statu quo outcome must be introduced.

[5] In the literature, the cake is often referred to by the interval [0, 1]. Here, we refer to the cake by CAKE to stress that we will not require in the sequel the utility function to be additive, which would be the case in the standard setting.

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