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A mixed beam model with non-uniform warpings derived from the Saint Venànt rod

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ABSTRACT

A new linear model for beams with compact or thin-walled section is presented. The formulation is based on the Hellinger–Reissner principle with independent descriptions of the stress and displacement fields. The kinematics is constituted by a rigid section motion and non uniform out-of-plane warpings related to shear and torsion. The stress field is built on the basis of the Saint-Venànt (SV) solution and with a new part to describe the variable warping.

The formulation of a finite element with exact shape functions made possible to validate the beam model avoiding discretization errors.

1. Introduction

3D beams with thin-walled or compact section are widely used in engineering practice and the improvement of both continuum models and FEM solution procedures represents a primary task for researchers.

Saint-Venànt (SV) rod theory, see [1], is a powerful theoretical basis for deriving beam models to be used in standard 3D frame analysis [2,3] because it allows an accurate one-dimensional description of the 3D continuum behaviour in terms only of cross-section generalized parameters. Subsequent extensions, like that of Ieşan [1], allows the SV solution also to be exploited for non-isotropic and non-homogeneous materials. In some cases of loading and boundary conditions, for example torsion actions applied to open profiles, the structural behaviour is not correctly described by the SV theory. In particular the end effects due to variable warping along the beam axis could produce important additional normal and shear stresses which are not negligible. Following the pioneering work of Vlasov [4] much researches has been devoted to the formulation of mechanical models capable of describing this phenomenon accurately. The initial theory has been notably refined in terms of both the theoretical aspects, for example see [5,6], and the numerical methods of analysis [7–10]. The main part of these works is focused on the analysis of thin-walled profiles which can be reasonably described using the sectorial areas theory, while more recent contributions extended the range of application by proposing beam theories suitable for the FEM analysis of one-dimensional structures with generic cross-sections and subjected also to non-uniform shear warping effects [11,12].

In the present work a linear beam model to account for the variable warping due to shear and torsion is derived. Its main feature is the way it maintains all the details of the exact SV solution in order to analyze beams with generic sections. This goal is reached by formulating the model on the basis of the Hellinger–Reissner variational principle, in which both the stress and displacement fields are described independently. With respect to beam theories derived only on the basis of kinematical hypotheses, the mixed formulation proposed allows a better evaluation of some 3D effects recovering, as subcase, the standard SV behaviour exactly. In particular, the kinematical description maintains, as other compatible models [12,8], a rigid section motion and out-of-plane deformations represented by the SV shears and torsion warping functions independently amplified along the beam axis. The stress field is more accurately evaluated as the sum of the exact contribution due to the SV solution and to some further terms due to variable warping. The assumed fields are introduced in the Hellinger–Reissner functional to obtain an accurate Ritz–Galerkin approximation of the beam model in terms of generalized static and kinematic quantities.

The work is focused on isotropic beams, however the SV solution is formulated in a way that allows, following the Ieşan approach [1], a possible extension to more general cases such as, for example, sections composed of several materials. The description of the stress field induced by the non-uniform warping is obtained on the basis of two distinct approaches which differ in the evaluation of the shear contributions. Both approaches account for the contribution of the shear stresses due to variable warping neglected in standard formulations based on Vlasov assumption. The first one uses a *Benscoter*-like [6] expression, deriving the

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stresses in a way similar to the kinematical description used in [12]. The second approach, denoted as *Jourawsky*-like, evaluates this contribution through the equilibrium equation in the axial direction, requiring the evaluation of 3 additional warping functions on the cross section domain.

The models proposed are sufficiently accurate but simple enough for use in technical applications, avoiding the onerous section analysis of more sophisticated formulations currently used for composite beam [13–15]. These ones are potentially very accurate but require major computational effort due, for example, to the solution of a quadratic eigenvalues problem. On the contrary the proposed formulation only requires the evaluation of a few warping functions that can be performed using a FEM approach like those presented in [2,3] (see also [16,11] for a solution based on the Boundary Element Method). The section compliance matrix, recovered from the stress field obtained, accounts for all the coupling effects arising from the 3D problem and is valid for the largest variety of cross sections (compact, thin/thick walled ones). Note that the mechanical properties of the cross section (principal flexural directions, flexural, torsional and bimoment inertia, shear center, etc.) always have to be preliminarily calculated. Their evaluation using simplified solutions specialized for thin walled beams are unnecessary because, as discussed in detail in [2], the computational cost required for a numerical evaluation of the warping functions is limited even when fine meshes are used.

The validation of the proposed models is performed by means of a mixed finite element formulated on the basis of exact shape functions. In particular the static fields interpolation exactly satisfy the homogeneous form of the equilibrium equations adopting an exponential distribution of the bi-moments and bi-shears, while the resultant force and moment are constant and linear respectively, as in standard beam models. The static interpolation also allows the discrete form of the strain energy to be evaluated exactly without using any explicit displacement interpolations. Externally the element exposes kinematical parameters only, thanks to the use of static condensation so reducing the global computational efforts. The finite element proposed has no discretization error. This feature allows us to perform the numerical experimentation focusing attention only on the beam model approximation. The numerical tests presented regard single or framed beam structures and the results obtained are compared with those proposed by other authors or calculated by using shell or solid FEM analyses.

As a final comment observe how the mixed model adopted here is particularly suitable for the extension to geometrically nonlinear analyses using corotational strategies [17–20] where displacements and rotations require complex change-of-observer rules on the contrary the stresses are not affected by this change. The use of a formulation valid for generic cross sections, that refers all the variables to the same axis and which is able to detect eventual coupling between torsional and shear warping, is a further advantage especially in the geometrically nonlinear case [21,20].

2. The generalized SV solution

In this section the standard SV solution is briefly described in a form which is easy to frame in the approach proposed by Ieşan [1] whose main advantage consists in the possibility of also analyzing inhomogeneous and anisotropic materials. For a review of the classical SV problem we refer also to [3,2].

2.1. The standard SV solution

Let us consider a cylinder occupying a reference configuration $\mathcal{B}$ of length ℓ confined by the lateral boundary (the so called *mantel*) denoted by $\partial\mathcal{B}$ and two terminal bases Ω_0 and Ω_ℓ on which the external forces are applied.

The cylinder is referred to a Cartesian frame $(\mathcal{O}, x_1, x_2, x_3)$ with unit vectors $\{\boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{e}_3\}$ and $\boldsymbol{e}_1$ aligned with the cylinder axis. From now on we assume the reference system aligned with the principal axes of inertia of the cross-section and its origin located in the centroid. In this system, see Fig. 1, we denote with $\boldsymbol{X}$ the position of a point P

$$\boldsymbol{X} = \boldsymbol{X}_0 + \boldsymbol{x} \quad \text{with} \quad \begin{cases} \boldsymbol{X}_0 = s\boldsymbol{e}_1 \\ \boldsymbol{x} = x_2\boldsymbol{e}_2 + x_3\boldsymbol{e}_3 \end{cases}$$

where $\boldsymbol{X}_0$ represents the position of P with respect to the beam axis, s is an abscissa which identifies the generic cross-section $\Omega[s]$ of the beam and $\boldsymbol{x}$ is the position of P inside $\Omega[s]$.

The displacement field $\boldsymbol{u}[\boldsymbol{X}]$ of the standard SV solution can be expressed, apart from an inessential rigid displacement field, as a function of 6 coefficients b_i in the following form:

$$\boldsymbol{u}[\boldsymbol{X}] = \boldsymbol{u}_0[s] + \boldsymbol{\varphi}[s] \wedge \boldsymbol{x} + \boldsymbol{u}_\omega[s, \boldsymbol{x}] \tag{1a}$$

where $\wedge$ denote the cross product and

$$\boldsymbol{u}_0[s] = \begin{bmatrix} sb_1 \\ -\frac{1}{6}s^3 b_6 - \frac{1}{2}s^2 b_3 \\ -\frac{1}{6}s^3 b_5 - \frac{1}{2}s^2 b_2 \end{bmatrix}, \quad \boldsymbol{\varphi}[s] = \begin{bmatrix} sb_4 \\ -u_{03,s} \\ u_{02,s} \end{bmatrix}.$$

u_{02} and u_{03} are the second and third components of $\boldsymbol{u}_0$ and the comma indicates the derivative operation. The function $\boldsymbol{u}_\omega[s,\boldsymbol{x}]$ assumes the following expression

$$\boldsymbol{u}_\omega[s, \boldsymbol{x}] := \sum_{i=1}^{6} \bar{b}_i[s]\boldsymbol{u}_\omega^{(i)}[\boldsymbol{x}], \quad \bar{b}_i[s] = b_i + s\delta_{2i}b_5 + s\delta_{3i}b_6 \tag{1b}$$

δ_{ij} being the Kroneker delta and

$$\boldsymbol{u}_\omega^{(1)} = \begin{bmatrix} 0 \\ -\nu x_2 \\ -\nu x_3 \end{bmatrix} \quad \boldsymbol{u}_\omega^{(2)} = \begin{bmatrix} 0 \\ -\nu x_2 x_3 \\ \nu\frac{x_2^2 - x_3^2}{2} \end{bmatrix} \quad \boldsymbol{u}_\omega^{(3)} = \begin{bmatrix} 0 \\ \nu\frac{x_3^2 - x_2^2}{2} \\ -\nu x_2 x_3 \end{bmatrix} \quad \boldsymbol{u}_\omega^{(k)} = \begin{bmatrix} w_1^{(k)}[\boldsymbol{x}] \\ 0 \\ 0 \end{bmatrix}. \tag{1c}$$

Note how $\boldsymbol{u}_\omega^{(i)}$ $(i = 1 \ldots 3)$ represent the in–plane deformations due to Poisson effects and $\boldsymbol{u}_\omega^{(k)}$ $(k = 4 \ldots 6)$ are the shear and torsional out–of–plane warpings.

The components $w_1^{(k)}$ $(k = 4 \ldots 6)$ will be denoted from now on simply as ω_1, ω_2 and ω_3, and collected in the vector $\boldsymbol{\omega}[\boldsymbol{x}] = \{\omega_1, \omega_2, \omega_3\}$. They are evaluated by means of the equilibrium equations once the strain $\boldsymbol{\varepsilon}$ has been obtained from the compatibility condition and the stress $\boldsymbol{\sigma}$ from the elastic constitutive laws

$$\begin{cases} \boldsymbol{\omega}_{,22} + \boldsymbol{\omega}_{,33} + 2\boldsymbol{x} = \boldsymbol{0} \in \Omega \\ n_2\boldsymbol{\omega}_{,2} + n_3\boldsymbol{\omega}_{,3} = \boldsymbol{g} \in \partial\Omega \end{cases}, \quad \boldsymbol{g} = \begin{bmatrix} x_3 n_2 - x_2 n_3, \\ \nu\left(\frac{x_2^2 - x_3^2}{2} n_2 + x_2 x_3 n_3\right) \\ -\nu\left(\frac{x_2^2 - x_3^2}{2} n_3 - x_2 x_3 n_2\right) \end{bmatrix}, \tag{2}$$

where $\boldsymbol{n} = \{0, n_2, n_3\}$ the direction normal to $\partial\mathcal{B}$.

Letting $\boldsymbol{s} = \boldsymbol{\sigma}\boldsymbol{e}_1 = \{\sigma_{11}, \sigma_{12}, \sigma_{13}\}$ be the traction applied on the generic cross section the six constants b_i are evaluated in terms of the beam section resultant force $\boldsymbol{N}[s]$ and moment $\boldsymbol{M}[s]$

$$\boldsymbol{N}[s] = \int_\Omega \boldsymbol{s}\,dA, \quad \boldsymbol{M}[s] = \int_\Omega \boldsymbol{x} \wedge \boldsymbol{s}\,dA. \tag{3}$$

The solution is defined apart from a self equilibrated stress state which depends on the exact force distribution on the end bases.

From now on, to simplify the notation, the dependence of the quantities on s and $\boldsymbol{x}$ will be omitted when clear from the context.