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A stress improvement procedure

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ABSTRACT

In this paper, we present a novel procedure to improve the stress predictions in static, dynamic and nonlinear analyses of solids. We focus on the use of low-order displacement-based finite elements – 3-node and 4-node elements in two-dimensional (2D) solutions, and 4-node and 8-node elements in 3D solutions – because these elements are computationally efficient provided good stress convergence is obtained. We give a variational basis of the new procedure and compare the scheme, and its performance, with other effective previously proposed stress improvement techniques. We observe that the stresses of the new procedure converge quadratically in 1D and 2D solutions, i.e. with the same order as the displacements, and conclude that the new procedure shows much promise for the analysis of solids, structures and multiphysics problems, to calculate improved stress predictions and to establish error measures.

1. Introduction

During the last decades, many different stress improvement procedures have been explored [1–27]. The aim is to reach enhanced stress predictions, as part of the solution of the mathematical models, and to establish solution error estimates [3,4]. If an effective scheme to enhance the stress predictions were available, the finite element method could be used with coarser meshes, reducing the expense of analysis. Furthermore, an effective scheme to assess the error would be valuable to assure an adequate solution. Early procedures were based either on stress smoothing [5,6] or L_2 projection techniques [7]; however, these approaches are not particularly effective and they have hardly been used in practice.

Considering inexpensive solution error indicators, the stress band plots proposed by Sussman and Bathe [1,8–10] have been used extensively, both for linear and nonlinear analyses, but of course these only give an indication of the solution accuracy – they do not improve the stress predictions.

The calculation of improved stress predictions is particularly important if low-order elements are to be used. For example, considering three-dimensional (3D) solutions, the use of 4-node constant strain tetrahedral elements would frequently be computationally efficient if the stresses could be predicted to a higher accuracy than given directly by the displacements. That is, the constant stress assumption, implied by the assumed linear displacements, is not good in many analyses.

A widely-recognised contribution towards a stress improvement procedure was published by Zienkiewicz and Zhu, when they proposed the 'superconvergent patch recovery' method [11]. This technique is based on the existence of superconvergent points, also referred to as Barlow points [12], where the stresses are of one order higher accuracy than at any other point in the element domain. Appropriate order polynomials approximating the stresses are smoothly fitted through these points, sometimes in a least squares sense. Later, variants of the original method were developed to further enhance its performance [13–15].

Although the superconvergent patch recovery methods seemed to work relatively well for certain elements, superconvergent points do not always exist – e.g. in triangular elements, distorted isoparametric elements and in elements with varying material properties (hence nonlinear analyses) – see the discussion by Hiller and Bathe [16]. Three widely used procedures that do not require the knowledge of superconvergent points are the 'posterior equilibrium method' (PEM), the 'recovery by equilibrium in patches' (REP) method, and the 'recovery by compatibility in patches' (RCP) method.

The PEM was proposed by Stein and Ohnimus [17] and is based on the work published earlier by Stein and Ahmad [18,19]. This method uses the principle of virtual work to calculate improved interelement tractions for the purposes of local error estimation [17,20]. The REP method was proposed by Boroomand and Zienkiewicz [21,22]. This method uses the principle of virtual work to calculate improved stresses within the finite element domain. The RCP method was proposed by Ubertini [23] and further developed by Benedetti et al. [24]. This method uses the principle of minimum complementary energy to calculate improved stresses that satisfy point-wise equilibrium. Later, Castellazzi et al.

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established a solution error estimate based on the RCP method to guide adaptive meshing [25].

All three stress calculation procedures yield impressive results that exceed the performance of the superconvergent patch recovery method. However, to ensure a well-posed problem for the solution of the unknown stress coefficients, several assumptions are employed, and these assumptions limit the accuracy of the results. Specifically, the PEM assumes that the improved interelement tractions are approximately equal (by a difference minimization) to the tractions directly-calculated from the displacement solution [17]; the REP method uses element nodal point forces that correspond to individual stress components [22]; and the RCP method imposes differential equilibrium for all points in the element [24], a constraint which is too severe, as a result the RCP solution is not reliable for all classes of problems.

Recently, we proposed the NPF-based method [26,27]. This procedure also employs the principle of virtual work, but without the assumptions used in the earlier methods. While the numerical results in Refs. [26,27] are encouraging, the method still requires to consider specific element stress domains and some stress averaging. We concluded, see Refs. [26,27], that a variational basis was necessary to obtain further insight and possibly improve the schemes.

For various problems in engineering and the sciences – like in the analysis of (almost) incompressible media, thin structures, and multiphysics phenomena – optimal finite element discretisations can only be obtained if mixed variational formulations are used [1,28–34]. Indeed, in Ref. [35], Mota and Abel show that the stress smoothing, L_2 projection and superconvergent patch recovery techniques are based on the use of the Hu-Washizu principle.

Our objective in this paper is to show that the PEM and the REP, RCP, and NPF-based methods are also all based, with certain assumptions, on the Hu-Washizu variational principle, and then present a novel and significantly improved procedure for stress predictions. Throughout we focus on the use of low-order displacement-based finite element discretisations of solids, that is, 2-node elements in 1D solutions, 3-node triangular and 4-node quadrilateral elements in 2D solutions, and 4-node tetrahedral and 8-node brick elements in 3D solutions. These elements are computationally efficient provided good stress convergence is obtained.

We analyse the new stress prediction procedure in detail for 1D problems using 2-node elements with arbitrary loading and material properties (but constant cross-sectional area), and prove that the procedure is reliable, giving stresses that are, in fact, optimal stress predictions (in the norm used), with the order of convergence being quadratic, i.e. the same order as for the displacements. This order of stress convergence is also seen numerically in 1D and 2D solutions. In a study, we compare the performance of the new method with the other above-mentioned procedures (that is, with the best stress improvement procedures currently available). It is important to note that we consider static, dynamic and nonlinear solutions. Throughout the paper we use the notation of Ref. [1].

2. Fundamental equations

Consider the equilibrium of a body of volume V and surface area S, subjected to externally applied surface tractions $\underline{f}^S$ on the area S_f and body forces $\underline{f}^B$; see Fig. 1. The body is supported on the area S_u with prescribed displacements $\underline{u}_p$, and, for now, linear analysis conditions are assumed. We seek to calculate the unknown displacements, strains and stresses.

In the differential formulation of the problem, the unknown response is calculated by solving the governing differential equations of equilibrium and compatibility, with the constitutive relationships, subject to the applied boundary conditions. That is, we solve

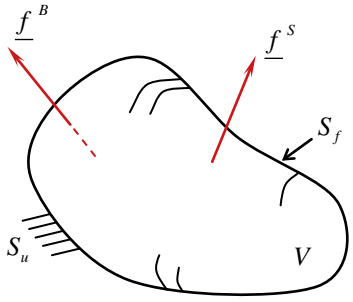

Fig. 1. General 3D body of volume V and surface area S, where $S_u \cup S_f = S$ and $S_u \cap S_f = 0$.

$$div[\underline{\tau}_{ex}] + \underline{f}^B = \underline{0}$$

$$\underline{\varepsilon}_{ex} = \partial_\varepsilon \underline{u}_{ex}$$

$$\underline{\tau}_{ex} = \underline{C}\ \underline{\varepsilon}_{ex}$$

subject to

$$\underline{u}_{ex} = \underline{u}_p \quad on\ S_u$$
$$\underline{f}^S = \underline{\tau}_{ex}\underline{n} \quad on\ S_f$$

where $\underline{u}_{ex}$, $\underline{\varepsilon}_{ex}$ and $\underline{\tau}_{ex}$ are the exact displacements, strains and stresses, respectively, ∂_ε is the differential operator on $\underline{u}_{ex}$ to obtain the strain components $\underline{\varepsilon}_{ex}$, $\underline{C}$ is the stress–strain matrix, and $\underline{n}$ is the unit outward normal vector on the surface S_f.

A second (but entirely equivalent) approach to the solution of the problem is given by minimising the total potential energy $\Pi(\underline{u})$,

$$\Pi(\underline{u}) = \int_V \frac{1}{2}\underline{\varepsilon}^T\underline{\tau}\ dV - \int_{S_f}\underline{u}^T\underline{f}^S\ dS - \int_V \underline{u}^T\underline{f}^B\ dV \tag{1}$$

with the constraints

$$\begin{aligned}&\underline{\varepsilon} = \partial_\varepsilon \underline{u}\\ &\underline{\tau} = \underline{C}\ \underline{\varepsilon}\\ &\underline{u} = \underline{u}_p\ on\ S_u\end{aligned} \tag{2}$$

where $\underline{u}$ is any displacement field satisfying the boundary condition on S_u, and $\underline{\varepsilon}$ and $\underline{\tau}$ are the strains and stresses corresponding to $\underline{u}$.

For approximate solutions, a larger class of trial functions can be employed when we operate on the total potential energy rather than on the differential formulation of the problem; see Refs. [1,10]. This has important consequences and much of the success of the finite element method hinges on this fact.

3. Finite element methods for stress predictions

In this section, we first review the displacement-based finite element method, then we present a mixed formulation based on the Hu-Washizu principle. Thereafter, we specialise this mixed formulation to arrive at the basic equations of the PEM and the REP, RCP, and NPF-based methods. Finally, we use this mixed formulation – and its properties – to present our new stress prediction scheme.

3.1. Displacement-based finite element method

In the displacement-based finite element method, we assume a displacement pattern within each element m, that is, $\underline{u}^{(m)} = \underline{H}^{(m)}\widehat{\underline{U}}$, where $\underline{H}^{(m)}$ is the displacement interpolation matrix and $\widehat{\underline{U}}$ lists the nodal point displacements of the assemblage (including those at the supports).

With this assumption, the strains $\underline{\varepsilon}^{(m)}$ and stresses $\underline{\tau}_h^{(m)}$ of element m follow directly from Eq. (2),