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Dynamic stiffness formulation and free vibration analysis of a spinning composite beam

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Abstract

The dynamic stiffness matrix of a spinning composite beam is developed and then used to investigate its free vibration characteristics. Of particular interest in this study is the inclusion of the bending–torsion coupling effect that arises from the ply orientation and stacking sequence in laminated fibrous composites. The theory is particularly intended for thin-walled composite beams and does not include the effects of shear deformation and rotatory inertia. Hamilton's principle is used to derive the governing differential equations, which are solved for harmonic oscillation. Exact expressions for the bending displacement, bending rotation, twist, bending moment, shear force and torque at any cross-section of the beam, are also obtained in explicit analytical form. The dynamic stiffness matrix, which relates the amplitudes of loads to those of responses at the end of the spinning beam in free vibration is then derived by imposing the boundary conditions. This enables natural frequency calculation of a spinning composite beam at various spinning speeds to be made by applying the Wittrick–Williams algorithm to the resulting dynamic stiffness matrix. The spinning speed at which the fundamental natural frequency tends to zero is the critical speed, which is established for a composite shaft that has been taken from the literature as an example. The results are discussed and some are compared with published ones. The paper concludes with some remarks.

Keywords: Dynamic stiffness method; Free vibration; Spinning composite beam; Critical spinning speed; Wittrick–Williams algorithm

1. Introduction

Many authors have investigated the free vibration behaviour of a spinning metallic beam (see Refs. [1–3] which provide further references on the subject) whereas that of a spinning composite beam made of laminated fibrous composites is relatively a recent research topic [4–9]. It is evident from the literature that the majority of published papers on the free vibration analysis of spinning metallic beams have used classical theory of differential equations which couple flexural motions in both principal planes, but ignores the torsional deformation. This is a reasonable assumption for many metallic beams, particularly with doubly symmetric cross-section for which the shear centre and centroid are coincident and as a consequence, the torsional motion will be uncoupled with the flexural ones. However, when dealing with the free vibration analysis of spinning composite beams, this assumption is not generally true and the complexity of the problem increases considerably, making the investigation more difficult. The difficulty arises because unlike metallic beams, composite beams, even with doubly symmetric cross-sections, exhibit material coupling between various modes of deformation caused by ply orientations. In particular, the coupling between the bending displacements and torsional rotation is of great significance in structural design.

The research in the area of spinning composite beams is specially driven by helicopter and automobile industries to develop lightweight drive shafts constructed from fibre-reinforced composite materials. Bert [6] appears to be one of the early investigators who presented a simple method for critical speed analysis of composite drive shafts by

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incorporating coupled bending–torsion composite beam theory. Later Kim and Bert [7] used a more accurate shell theory for the composite shaft and made a direct comparison of results with those obtained earlier from the beam theory [6]. Song et al. [8] on the other hand investigated both vibration and stability behaviour of an elastically tailored rotating shaft using beam theory. Their investigation placed particular emphasis on the effects of conservative and gyroscopic forces on the result. Song et al. [9] subsequently developed a more advanced anisotropic beam theory to study both vibration and stability control of smart composite rotating shafts by using structural tailoring and piezoelectric strain actuation techniques. The present paper however, uses a different approach and develops the dynamic stiffness matrix of a spinning composite beam by including the bending–torsion coupling effect and then uses it to investigate its free vibration characteristics. Following the authors' recent work on the dynamic stiffness formulation and free vibration analysis for a spinning metallic beam [3], it is highly pertinent and timely to extend their earlier theory to the important case of composites.

The investigation is carried out in following steps.

1. The governing differential equations of motion of a spinning composite beam in free vibration are derived using Hamilton's principle. The coupling effect between the bending and torsional displacements is fully taken into account when deriving the theory.
2. For harmonic oscillation, the governing differential equations are solved in closed analytical form for bending displacements, bending rotations and twist.
3. Expressions for shear forces, bending moments and torque are also obtained explicitly from the solutions of the governing differential equations.
4. Next the shear forces, bending moments and torque are related to the bending displacements, bending rotations and twist by recasting the expressions for loads and responses and linking them through the dynamic stiffness matrix relationship.
5. The Wittrick–Williams algorithm [10] is then used as a solution technique to compute the natural frequencies, mode shapes and critical spinning speeds of an illustrative example.
6. Numerical results are given and discussed and this is followed by some concluding remarks.

2. Theory

2.1. Derivation of the governing differential equations

Fig. 1 shows a uniform circular spinning beam made of laminated composites, in a right-handed rectangular Cartesian coordinate system. The beam has a length L, mass per unit length m, polar mass moment of inertia per unit length I_α. The principal axes bending rigidities are EI for both

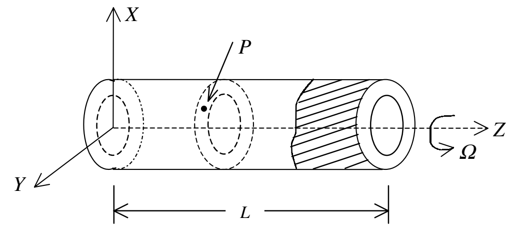

Fig. 1. A spinning composite beam.

planes, the torsional rigidity is GJ, and K is the bending–torsion coupling rigidity. Determination of EI, GJ and K for a composite beam has been addressed in the literature in numerous papers, for example see Refs. [11,12]. (For further details on how the cross-sectional and other properties of an anisotropic composite beam can be related through the expressions for strains and material properties, the works of Hodges et al. [13] and Giavotto et al. [14] are recommended.) The beam shown in Fig. 1 is spinning about the Z-axis with a constant angular velocity Ω in rad/s.

At a cross-section z from the origin, u and v are displacements of a point P in the X and Y directions, respectively, and the cross-section is allowed to rotate or twist about OZ by $\phi(y,t)$, so that the position vector $\mathbf{r}$ of the point P after deformation is given by

$$\mathbf{r} = (u - \phi y)\mathbf{i} + (v + \phi x)\mathbf{j} \tag{1}$$

where $\mathbf{i}$ and $\mathbf{j}$ are unit vectors in the X and Y directions, respectively.

The velocity of the point P is thus given by

$$\mathbf{v} = \dot{\mathbf{r}} + \boldsymbol{\Omega} \times \mathbf{r} \tag{2}$$

where $\boldsymbol{\Omega} = \Omega\mathbf{k}$, with $\mathbf{k}$ as the unit vector in the Z direction.

Substituting Eq. (1) into Eq. (2) and noting that $\mathbf{k}\times\mathbf{i} = \mathbf{j}$ and $\mathbf{k}\times\mathbf{j} = -\mathbf{i}$, give

$$\mathbf{v} = \{(\dot{u} - \dot{\phi}y) - \Omega(v + \phi x)\}\mathbf{i} + \{(\dot{v} + \dot{\phi}x) + \Omega(u - \phi y)\}\mathbf{j} \tag{3}$$

The kinetic energy T is given by

$$T = \frac{1}{2}\rho \int\int_V\int |\mathbf{v}|^2 \,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z = \frac{1}{2}\rho\int_0^L\int_A \mathbf{v}\cdot\mathbf{v}\,\mathrm{d}A\,\mathrm{d}z \tag{4}$$

Substituting $\mathbf{v}$ from Eq. (3) into Eq. (4) and noting that $\int_A x\,\mathrm{d}A = \int_A y\,\mathrm{d}A = 0$, $m = \rho A$ and $I_\alpha = \rho\int_A(x^2+y^2)\,\mathrm{d}A$, T can be expressed as

$$T = \frac{1}{2}m\int_0^L[\dot{u}^2 + \dot{v}^2 + 2\Omega(u\dot{v} - \dot{u}v) + \Omega^2(u^2+v^2)]\,\mathrm{d}z + \frac{1}{2}I_\alpha\int_0^L[\dot{\phi}^2 + \Omega^2\phi^2]\,\mathrm{d}z \tag{5}$$

The potential (or strain) energy U can be obtained using the procedure outlined in Refs. [8,9] and is given by