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Free vibration of beams carrying spring-mass systems – A dynamic stiffness approach

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ABSTRACT

Free vibration analysis of beams carrying spring-mass systems is carried out by using the dynamic stiffness method. The eigenvalue problem for the free vibration study is formulated by assembling the dynamic stiffness matrices of beam and spring-mass elements. The Wittrick–Williams algorithm is then applied to yield the required natural frequencies and mode shapes of the combined system. Numerical examples are given for a cantilever beam carrying a spring-mass system at the tip. A parametric study is then carried out by varying the mass and stiffness properties of the spring-mass system and the subsequent effects on the natural frequencies and mode shapes are illustrated. The proposed theory can be applied for other boundary conditions of the beam and can be extended to complex structures carrying spring-mass systems. The results are discussed and validated against published literature.

1. Introduction

Investigation into the free vibration behaviour of beams carrying spring-mass systems has continued to inspire researchers over the years. The literature on the subject is vast, but a sample of recently published papers [1–23] given in the list of references provides sufficient technical details and good cross-references on the subject. The importance of this research is well recognized because of its engineering applications. For instance, in the design of robotics and also when dealing with human structure interaction, structural systems are often modelled as a combination of beam and spring-mass systems [14,17]. Furthermore, the design of tuned vibration absorber relies on this concept that by adding one or more spring-mass systems to a structure, the natural frequencies and mode shapes can be altered in a significant way so as to avoid resonance and other undesirable dynamic phenomena. This is extremely useful from a stability point of view, particularly when dealing with aeroelastic problems of aeronautical structures. The main features of some selective references are reviewed next.

Chang [1] investigated the free vibration characteristics of a simply supported beam carrying a heavy concentrated mass at the centre. He included the effect of rotatory inertia for both the beam and the concentrated mass in his analysis, but there was no consideration of spring attachments. Low [2] also did not include spring attachments, but he used multiple masses attached to the beam at different locations when deriving the frequency equation of the combined system. He used both frequency determinant and Laplace transformation methods and showed that the latter was computationally more demanding than the former. Cha [3] determined the natural frequencies of linear elastica which was essentially a beam carrying a number of spring-mass systems. An interesting feature of his work was that he manipulated the characteristic determinant associated with the generalized eigenvalue problem and reduced the problem to a smaller size which was computationally efficient. Rossit and Laura [4,5] studied the free vibration problem of a cantilever beam with a spring-mass system attached at the tip. They obtained exact solution for a wide range of beam and spring-mass parameters, but unlike the present paper, without resorting to the dynamic stiffness method. Their solution being exact will be extensively used to validate the present theory. Wu and Chen [6] on the other hand used a numerical assembly technique and avoided derivation of lengthy mathematical expression for the frequency equation when investigating the free vibration behaviour of Timoshenko beams carrying multiple spring system. Naguleswaran [9] examined the transverse vibration of a Bernoulli–Euler beam carrying several particles. He developed a systematic procedure to compute progressively the elements of the frequency determinant by using a recurrence relationship to arrive at the frequency equation. Chen and Wu [10] used Bessel functions and obtained exact natural frequencies and mode shapes of non-uniform beams with attached multiple spring-mass systems. Low [11] concluded that Rayleigh's approximate method based to assumed deformed shape gives sufficiently accurate natural frequencies for a beam-mass system in a computationally efficient manner when compared with the exact

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solution. Zhou and Ji [16] proposed an interesting analysis which differed from previous studies in that they used continuously distributed spring-mass systems over a beam when investigating the free vibration characteristics. Their focus was to simulate human-structure interaction and they drew many useful conclusions. Lin and Tsai [18] took a step further by analysing multi-span beams carrying multiple spring-mass systems. For a certain class of problems, they compared the accuracy of numerical methods against exact results. Other notable contributors to this research in recent years, amongst many, are Wang et al. [19], Wu and Hsu [20], Magrab [21], Yesilce and Demirdag [22] and Tang et al. [23]. Different, but related research to alter the natural frequencies and mode shapes of beam, plate and shell structures using tuned mass damper and/or penalty function methods for various applications including optimal vibration control can be found in Refs. [24–30]. Given the above context, a dynamic stiffness method [31], generally known as DSM, is proposed in this paper for the first time to deal with the free vibration problem of beams carrying spring-mass systems. The element properties used in the DSM is based on the exact solution of its governing differential equation of motion in free vibration. Thus, the DSM with its much better model accuracy has many advantages over finite element and other approximate methods. It gives exact results that are independent of the number of elements used in the analysis. Furthermore, the DSM is sufficiently general like the finite element method (FEM) so that structural elements can be assembled, offset or rotated, but importantly, in sharp contrast to FEM, results from the DSM analysis are always exact. In essence the dynamic stiffness matrices of beam and spring-mass elements are developed and assembled in this paper to form the overall dynamic stiffness matrix of the combined system. The ensuing eigenvalue problem is solved by applying the well-known algorithm of Wittrick and Williams [32], yielding natural frequencies and mode shapes of the structure. A cantilever beam carrying a spring-mass system is analysed to serve as an example, but the theory developed can be applied in a much wider context. The results are validated against published literature.

2. The dynamic stiffness matrix of a beam and a spring-mass system

For a Bernoulli–Euler beam element, the dynamic stiffness matrix, relating the amplitudes of forces (moments) and displacements (rotations) at its nodes, is readily available [33]. Referring to the nodal forces (moments) and displacements (rotations) shown for flexural motion in Fig. 1, the dynamic stiffness matrix relationship of a beam with flexural rigidity EI, mass per unit length ρA, and length L, is given by

$$\begin{bmatrix} f_{y1} \\ m_1 \\ f_{y2} \\ m_2 \end{bmatrix} = \begin{bmatrix} d_1 & d_2 & d_4 & d_5 \\ d_2 & d_3 & -d_5 & d_6 \\ d_4 & -d_5 & d_1 & -d_2 \\ d_5 & d_6 & -d_2 & d_3 \end{bmatrix} \begin{bmatrix} \delta_{y1} \\ \theta_1 \\ \delta_{y2} \\ \theta_2 \end{bmatrix} \tag{1}$$

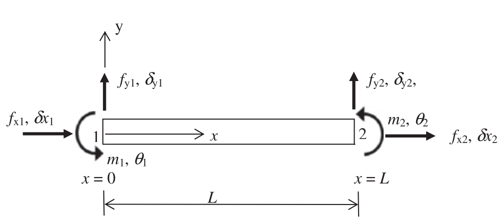

Fig. 1. Coordinate system and notation for forces (moments) and displacements (rotations) at nodes 1 and 2 of a Bernoulli–Euler beam.

or

$$\mathbf{f} = \mathbf{k}\delta \tag{2}$$

where $\mathbf{f}$ and δ are respectively the force (moment) and displacement (rotation) vectors and $\mathbf{k}$ is the frequency dependent 4×4 dynamic stiffness matrix whose elements $k(i,j)$ ($i = 1,2,3,4; j = 1,2,3,4$) are given by $d_1 - d_6$ as shown below. (Note that $\mathbf{k}$ is symmetric as expected.)

$$d_1 = \frac{W_3\lambda^3(S_\lambda C_{h\lambda} + C_\lambda S_{h\lambda})}{\Delta}, \quad d_2 = \frac{W_2\lambda^2 S_\lambda S_{h\lambda}}{\Delta},$$
$$d_3 = \frac{W_1\lambda(S_\lambda C_{h\lambda} - C_\lambda S_{h\lambda})}{\Delta} \tag{3}$$

$$d_4 = -\frac{W_3\lambda^3(S_\lambda + S_{h\lambda})}{\Delta}, \quad d_5 = \frac{W_2\lambda^2(C_{h\lambda} - C_\lambda)}{\Delta},$$
$$d_6 = \frac{W_1\lambda(S_{h\lambda} - S_\lambda)}{\Delta} \tag{4}$$

with

$$\lambda = \sqrt[4]{\frac{\rho A\omega^2 L^4}{EI}}, \quad W_1 = \frac{EI}{L}, \quad W_2 = \frac{EI}{L^2}, \quad W_3 = \frac{EI}{L^3} \tag{5}$$

$$C_\lambda = \cos\lambda, \quad S_\lambda = \sin\lambda, \quad C_{h\lambda} = \cosh\lambda, \quad S_{h\lambda} = \sinh\lambda, \quad \Delta = 1 - C_\lambda C_{h\lambda} \tag{6}$$

The amplitudes of forces and displacements of a harmonically vibrating spring mass system are shown in Fig. 2. The dynamic stiffness matrix relationship, assuming that the spring is mass-less, is given by

$$\begin{bmatrix} f_{y1} \\ f_{y2} \end{bmatrix} = \begin{bmatrix} k & -k \\ -k & k - \omega^2 m \end{bmatrix} \begin{bmatrix} \delta_{y1} \\ \delta_{y2} \end{bmatrix} \tag{7}$$

or

$$\mathbf{f} = \mathbf{k}\delta \tag{8}$$

Fig. 3 shows a cantilever beam carrying a spring-mass system at the tip. The connecting nodes 1 and 2 represent the beam whereas nodes 2 and 3 are connected by a spring-mass element. The dynamic stiffness matrices of the two elements can be assembled by using Eqs. (1) and (7) and by deleting the displacement and rotation components f_{y1} and θ_1 at the built-in node 1 to form the eigenvalue problem as follows:

$$\begin{bmatrix} d_1 + k & -d_2 & -k \\ -d_2 & d_3 & 0 \\ -k & 0 & k - \omega^2 m \end{bmatrix} \begin{bmatrix} f_{y2} \\ \theta_2 \\ f_{y3} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \tag{9}$$

or

$$\mathbf{KD} = 0 \tag{10}$$

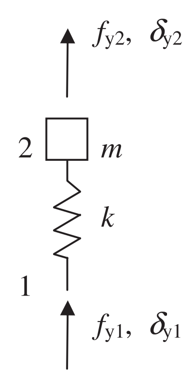

Fig. 2. Forces and displacements of a spring-mass system connecting nodes 1 and 2.